QUANTUM FIELD THEORY: THE WIGHTMAN AXIOMS AND THE HAAG-KASTLER AXIOMS

LOUIS-HADRIEN ROBERT

The aim of the Quantum field theory is to offer a compromise between quantum mechanics and relativity. The fact that quantum mechanics and relativity are not compatible can be easily derived from the following fact:

A relativistic 1-particle system with spin 0 is a solution of the equation:

$$i\frac{\partial}{\partial t}\varphi(x,t) = \sqrt{m^2 - \Delta^2}\varphi(t,x)$$

where φ is a function of time t and space x and Δ is the Laplace operator. In particular, when the initial value $x \mapsto \varphi(0,x)$ has compact support, then its time derivative does not vanish on any nonempty open set. This implies that the probability that a the particle travel faster that light is not 0. This is problematic

We may find different explanations on why this problem occurs, but one should agree that in a relativistic theory, space and type should be of the same type. This is not the case in *classical* quantum mechanics since position is an observable but time is not. There are two ways to solve the problem: we can either suppress the time dependence of the operators (this basically means inserting the time in the Hilbert space of states), or suppose that the operators have a space-time dependency. We consider the second option.

1. Wightman axioms

I will try to motivate Wightman axioms from my naive understanding of mathematician. We have three basic ingredients:

- the Minkowski space M,
- an Hilbert space $\mathcal{H}$,
- a 1-dimensional subspace of $\mathcal{H}$.

And a few basic physic intuitions:

- Observables are represented by self-adjoint operators on $\mathcal{H}$,
- If two observables do not interact one with another, the underlying opertors commute,
- The total energy of a system is bounded below (this comes from the fact that we would like the system to have a stable equilibrium). We can actually suppose up to a shift, that the total energy is non negative.

1.1. Minkowski space.

We denote by $M = (\mathbb{R}^4, (.,.))$, the vector space $\mathbb{R}^4$ with the Minkowski product:

$$(x,y) = x_0y_0 - (x_1y_1 + x_2y_2 + x_3y_3)$$

We will denote (x,x) by $||x||^2$ even if it does not have to be positive. We denote $\mathcal{P}$ (for Poincaré) the group of affine isometries which preserve time direction and space orientation (the normal subgroup of linear isometries (still preserving time direction and space orientation) is denoted by $\mathcal{L}$ for Lorentz). An element of $\mathcal{P}$ can be thought as a composition of a translation in space time followed by a Lorentz transformation (that is to say that $\mathcal{P}$ is a semi-direct product of $\mathcal{L}$ and the group of the translations which is isomorphic to M). For g in $\mathcal{P}$ and x in M, we denote $g \cdot x$ the image of x by the transformation g.

Definition 1.1. Two elements x and y of M are *space-like separated* (resp. *time-like separated*) if $||x - y||^2 < 0$ (resp. $||x - y||^2 > 0$).

1.2. **States and Hilbert space.** Just as for quantum mechanics, we have a set of states. It is supposed to be the projective space $\mathbb{P}(\mathcal{H})$ of some Hilbert space $(\mathcal{H}, \langle . | . \rangle)$ (that is the set of line (or ray) in $\mathcal{H}$). There is a subtlety here: a line d in $\mathcal{H}$ may be represented by a any non-zero vector x of d and we might write $d = [x]$. For any non-zero complex number we have $[x] = [zx]$. It would be nice to have for each line d in $\mathcal{H}$ a "canonical" representative element x of d in $\mathcal{H}$. We can for example ask x to be of norm 1, but this is not enough since for all real number θ, $[e^{i\theta}x]$ would represent the same line d. This causes problem to define a sum of states. However the "angle" or actually the cosine of the angle of two lines is well define: if $d_1 = [x_1]$ and $d_2 = [x_2]$ are two lines in $\mathcal{H}$, we set:

$$\langle d_1 | d_2 \rangle = \frac{| \langle x_1 | x_2 \rangle |^2}{\langle x_1 | x_1 \rangle \langle x_2 | x_2 \rangle} = \langle d_2 | d_1 \rangle \in [0, 1].$$

The idea of quantum mechanics is to interpret this quantity as follows: if d_1 is the eigenspace of some observable say O for an eigenvalue λ, and our system is prepared in the state d_2, then $\langle d_1 | d_2 \rangle$ is the probability that the result of the measurement by O of our system is λ.

We would like the *rules of physic* to be invariant by a changing of space-time frame: if the observer and the system change of space time frame we want that the result of the measurement satisfies the same law. Let us write down what this means. If g is an element of the Poincaré group and d is a state, there should exist another state $g \cdot d$ representing the state d in the space-time shifted by g and these "changing state" operations should be well-behaved: in particular we want $g \cdot (g' \cdot d) = gg' \cdot d$. We say that *the group $\mathcal{P}$ acts on the set of states.*

The preservation of the rules of physic means that we want for every element g in $\mathcal{P}$ and all states d_1 and d_2:

$$\langle g \cdot d_1 | g \cdot d_2 \rangle = \langle d_1 | d_2 \rangle \, .$$

This imposes a very rigid structure on how the group $\mathcal{P}$ intertwines the states: this actually induces a representation of $\overline{\mathcal{P}}$ on $\mathcal{H}$. Let us detail how this works:

Theorem 1.2 (Wigner)**.** *Let $u : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ be a map such that for all (d_1, d_2) in $\mathbb{P}(\mathcal{H})^2$, we have $\langle u(d_1) | u(d_2) \rangle = \langle d_1 | d_2 \rangle$. Then there exists a map $U : \mathcal{H} \to \mathcal{H}$ either anti-linear and anti-unitary or linear and unitary, such that for all x in $\mathcal{H}$, $u([x]) = [U(x)]$. Furthermore, two such maps are equal up to a multiplication by a complex number of norm 1. We say that U is a* lift *of u to $\mathcal{H}$.*

This means that for each g in $\mathcal{P}$, we can find a lift $U_g : \mathcal{H} \to \mathcal{H}$. For every g in $\mathcal{P}$, we can find h in $\mathcal{P}$ such that $g = h^2$. $U_h \circ U_h$ is certainly a lift of $(d \mapsto g \cdot d)$

and is linear, therefore $U_g = e^{i\alpha} U_h \circ U_h$ is linear. Hence for every g in $\mathcal{P}$, there exists a linear and unitary lift U_g. We would like to choose the U_g (remember that we can still multiply them by a complex number of norm 1) such that we have the following relation:

$$U_{gg'} = U_g \circ U_{g'}$$

This is not always possible because $\mathcal{P}$ is not simply connected (we actually have $\pi_1(\mathcal{P}) = \mathbb{Z}/2\mathbb{Z}$). That is why we need to work with the universal cover $\overline{\mathcal{P}}$ of $\mathcal{P}$. This is a "bigger[1] version" of the group $\mathcal{P}$.

From the previous discussion we deduce that we expect $\mathcal{H}$ to carry a unitary representation of $\overline{\mathcal{P}}$ the universal covering of the Poincaré group.

One state $\Omega = [\omega]$ in $\mathbb{P}(\mathcal{H})$ should represent the vacuum. Being the vacuum is invariant under changing of time-space frame. Hence we want that $\mathbb{C}\omega \subset \mathcal{H}$ to be a one-dimensional sub-representation of $\overline{\mathcal{P}}$. Quite often in the literature the state Ω is assumed to be the only one to fulfill this condition.

1.3. **The Poincaré group and its friends.** Similarly to $\mathcal{P}$, the group $\overline{\mathcal{P}}$ is a semi-direct product of M with the universal covering $\overline{\mathcal{L}}$ of $\mathcal{L}$. In particular M is a subgroup of $\overline{\mathcal{P}}$. It is worthwhile to note that this subgroup is commutative.

The time evolution of the system is contained in the representation of $\overline{\mathcal{P}}$ since the translations in time (actually in space as well) are elements of $\overline{\mathcal{P}}$. The expectations we should have about the time evolution does not deal with the long time but rather with infinitesimal time. Fortunately we have the following theorem:

Theorem 1.3 (Stone). *If ρ is a strongly continuous unitary representation of $\mathbb{R}$ in an Hilbert space W, then there exists an (unbounded) self-adjoint operator $A : W \to W$ with domain D such that for all t in $\mathbb{R}$ we have $\rho(t)_{|D} = \exp(itA)$.*

For x_0 in $\mathbb{R}$, let us denote by U_{x_0} the unitary operator corresponding to a translation of time by x_0. The previous theorem implies that we have a certain unbounded self-adjoint operator P_0 with domain D such that $(U_{x_0})_{|D} = \exp(ix_0P_0)$. In other words P_0 encodes the infinitesimal time evolution at the level of the Hilbert space $\mathcal{H}$. This operator should represent the energy for a physical reasons I cannot explain.

The same discussion about the space translations gives the existence of $-P_1$, $-P_2$ and $-P_3$. The operators P_0, P_1, P_2 and P_3 commute two by two because M is abelian. We may consider their joint spectrum $\sigma(P)$.

Definition 1.4. Let $Q_1, \dots, Q_r$ a collection of two by two commuting self-adjoint operators. The *joint spectrum* of $Q_1, \dots, Q_r$ is the set of element $(\lambda_1, \dots, \lambda_r)$ of $\mathbb{C}^r$ such that for all $(t_1, \dots, t_r)$ in $\mathbb{C}^r$, $\sum_{i=1}^r \lambda_i t_i$ is in the spectrum of $\sum_{i=1}^r t_i Q_i$.

As I said in the introduction, we want to impose the energy to be non-negative. Furthermore we want this condition to remain valid when we change of space-time frame. The non-negativity of the energy in all frame is equivalent to say that the joint spectrum of P_0, P_1, P_2, P_3 is in the positive light cone:

$$C^+ = \{x \in M | x_0 \geqslant 0 \text{ and } (x,x) \geqslant 0\}.$$

[1]We have a canonical map: $p : \overline{\mathcal{P}} \to \mathcal{P}$ which is a group morphism and is $2:1$.

That is the set of points in the space time which are time-like separate with 0 and happens after 0.

1.4. **Fields.** Just like in quantum mechanic, we have observables. Physically observables correspond to quantities which could be measured in a laboratory. The paradigm of quantum mechanics is that observables are represented by some self-adjoint operators on an Hilbert space. In the following sense: For some reasons, Suppose we have a system in a state d and that an observer measures an observable O. Suppose for sake of simplicity that there exist an orthonormal Hilbert-basis $(x_n)_{n\in\mathbb{N}}$ of eigenvectors of O associated to (real, because O is self-adjoint) eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$, suppose furthermore that all the eigenvalues are distinct. Then the probability that the measurement is λ_i is equal to $\langle [x_i] | d\rangle$. In particular if $d = [x]$, the real number[2]$\langle x|Ox\rangle \,/\, \langle x|x\rangle$ should be the average of the measurement if we iterate the experiment infinitely many times.

Of course here, our observables have to deal with the Minkowski space. The first guess is that a given measurement can be performed at different places and at different times. This means that we want *a priori* an observable O to be a function from M to the space of self-adjoint operators on $\mathcal{H}$. It turns out that this will not be a good definition. Let us such a function from $M \to \mathcal{O}(\mathcal{H})$ a mock-observable.

We would like "the rules of physic" to be invariant under changing of space-time frame. As we already discussed, changing of space-time frame is encoded by an element of $\mathcal{P}$ at the level of $\mathbb{P}(\mathcal{H})$, and by an element of $\overline{\mathcal{P}}$ at the level of $\mathcal{H}$. This is why we ask to have the following equality for all g in $\overline{\mathcal{P}}$, x in M, all y_1 and y_2 in $\mathcal{H}$ and all mock-observables O:

$$\left\langle U(g^{-1})y_1 | O(x)U(g^{-1})y_2\right\rangle = \langle y_1 | O(g\cdot x)y_2\rangle$$

As this should hold for every y_1 and y_2, we obtain (because $U(g^{-1}) = U^*(g)$):

$$U(g)O(x)U(g)^* = O(g\cdot x)$$

Let us consider the vacuum state Ω and O_1 and O_2 two mock-observables (or mock fields). Experiences show that we should have observables (or mock-observables) such that $\langle \Omega | [O_1(0_M), O_2(0_M)]\Omega\rangle \neq 0$.We can consider the function:

$$\begin{array}{rcl} t(O_1,O_2) : M & \to & \mathbb{C} \\ x & \mapsto & \langle \Omega | [O_1(x), O_2(0_M)]\Omega\rangle = \langle \Omega | [U_x O_1(0_M) U_x^*, O_2(0_M)]\Omega\rangle\,, \end{array}$$

where $U_x = \exp(ix_0P_0 - ix_1P_1 - ix_2P_2 - ix_3P_3)$. If we complexify the Minkowski space, we can write:

$$t(O_1,O_2)(x+iy) = \underbrace{\left\langle \Omega | O_1(0_M)U^*_{x+iy}O_2(0_M)\Omega\right\rangle}_{\text{real analytic on a big set}} - \underbrace{\left\langle \Omega | O_2(0_M)U_{x+iy}O_1(0_M)\Omega\right\rangle}_{\text{real analytic on another big set}}\,.$$

As we will see in a moment, we expect to have $[O_1(x), O_2(0_M)] = 0$ when x is space-like. This implies that the two above mentioned functions agree on a big set. A theorem on real analytic functions called the edge-of-the wedge theorem, would then imply that $\langle \Omega | [O_1(0_M), O_2(0_M)]\Omega\rangle = 0$.

This shows that we want the observables to be "generalized" function: in fact we want them to be tempered distribution: for every smooth function $f : M \to \mathbb{R}$ such

[2]It could be infinite.

that all its differentials decrease faster than any (inverse of) polynomials, (this set is denoted $\mathcal{S}(M)$, $\mathcal{S}$ is for Schwartz, the element of $\mathcal{S}(M)$ are called *test functions*) we have a self adjoint operator $O(f)$: An observable is a function:

$$O : \mathcal{S}(M) \to \mathcal{O}(\mathcal{H}).$$

It is worthwhile to note that the group $\overline{P}$ acts on $\mathcal{S}(M)$ via the following formula:

$$\begin{array}{rcl} g \cdot f : M & \to & \mathbb{R} \\ x & \mapsto & f(p(g^{-1}) \cdot x) \end{array}$$

where $p : \overline{\mathcal{P}} \to \mathcal{P}$ is the canonical projection from the universal covering of $\mathcal{P}$ on $\mathcal{P}$.

1.5. **Causality.** One of the paradigm of quantum mechanics can be sum up this way: if two measurements do not interact with each other, then the corresponding self-adjoint operators should commute. In the context of QFT this has a special implication since we want that no information travels faster than light.

Let us consider two test functions f_1 and f_2 whose supports are space-like separated. This means an information which would be shared by the two supports would have to travel faster than light. Let O_1 and O_2 be two observables. We want to have:

$$[O_1(f_1), O_2(f_2)] := O_1(f_1)O_2(f_2) - O_2(f_2)O_1(f_1) = 0$$

1.6. **Domain.** We want to be able to measure every combination of observable (maybe with uncertainty), hence we want to have a common dense domain D included in the domain of all observables and stable by the observables. Hence we actually want that the previous equality holds (at least) on D.

1.7. **The axioms.** We can now give Wightman axioms. A QFT consists of:

- an Hilbert space $(\mathcal{H}, \langle .|. \rangle)$,
- a subspace $\mathbb{C}\omega$ of $\mathcal{H}$ of dimension 1,
- a unitary representation of $\overline{\mathcal{P}}$ in $\mathcal{H}$ (denoted by $g \mapsto U(g)$),
- a collection $(O_i)_{i \in I}$ of operator-valued distributions on M with a common dense domain D which are self-adjoint (or symmetric) on their common domain.

such that:

Vacuum The space $\mathbb{C}\omega$ is a sub-representation of $\mathcal{H}$ (sometimes $\mathbb{C}\omega$ is required to be the only sub-representation of $\mathcal{H}$ of dimension 1).

Nice domain For every test function f and $i \in I$, $O_i(f)(D) \subseteq D$.

Equivariance For all g in $\overline{P}$, all i in I and all test function f:

$$U(g)O_i(f)U(g)^* = O_i(g \cdot f).$$

Causality For all $(i, j) \in I^2$ and all $(f_1, f_2) \in \mathcal{S}(M)^2$ with space-like separated support:

$$[O_i(f_1), O_j(f_2)]_{|D} = 0.$$

Spectrum The joint spectrum of the operators P_j (see section 1.3) is contained in C^+.

There are two last axioms. The first one makes sure that we are not considering an artificially too big Hilbert space:

Completeness We can approximate any operator on $\mathcal{H}$ by linear combinations of products of $(O_i(f))_{i\in I, f\in\mathcal{S}(M)}$. That is the sub-algebra of the algebra of operators on $\mathcal{H}$ generated by $(O_i(f))_{i\in I, f\in\mathcal{S}(M)}$ is dense.

The words "approximation" and "dense" have no precise meanings here, but we can phrase it in a rather algebraical way: we suppose that no closed subspace of $\mathcal{H}$ is stable by all the operators $(O_i(f))_{i\in I, f\in\mathcal{S}(M)}$.

Furthermore, we expect that some kind of physical law govern the time evolution of the universe. The last axiom says:

Time-slice axiom It is possible to achieve completeness when restricting the function $f \in \mathcal{S}(M)$ with domain contained in $\mathcal{U}_{\varepsilon,t} := \{x \in M \,||x_0 - t| < \varepsilon\}$ for an arbitrary small ε and an arbitrary time.

In other words, the initial conditions completely encodes the development of the system (and its backward development).

2. Haag–Kastler axioms

From the Wightman axioms, we can construct for each open subset $\mathcal{U}$ of M an algebra $\mathcal{A}(\mathcal{U})$: this is the subalgebra of $\mathcal{O}(\mathcal{H})$ generated by the operators $O_i(f)$ where f is any smooth function which has support included in $\mathcal{U}$. An element of $\mathcal{A}(\mathcal{U})$ is a polynomial in the variable $O_i(f)$. In the framework of the Wightman axioms, the operators $O_i(f)$ are typically not bounded. In the Haag–Kastler axioms, we suppose that these operators are all bounded. This means that this algebra is endowed with a norm. We may ask how the observation we made before can be translated in this new context.

We now forget (in the formalism, not for the intuition) about the observables $O_i(f)$ and we consider the following abstract data (which is called *net of algebras*):

$$\mathcal{U} \mapsto \mathcal{A}(\mathcal{U})$$

We still want to think of $\mathcal{A}(\mathcal{U})$ to act naturally on a certain Hilbert space. Therefore, we should have a notion of complex conjugate: We want the algebras $\mathcal{A}(\mathcal{U})$ to be endowed with an involution $* : \mathcal{A}(\mathcal{U}) \to \mathcal{A}(\mathcal{U})$ compatible with the structure of $\mathbb{C}$-vector spaces such that $||xx^*|| = ||x||^2$. This is were we need the operators to be thought as bounded. In mathematical terms, this means that the algebras $\mathcal{A}(\mathcal{U})$ are C-$*$-algebras.

If $\mathcal{U}_1 \subseteq \mathcal{U}_2$, a function with support included in $\mathcal{U}_1$ has support included in $\mathcal{U}_2$. From this we deduce, that we would like to have for every inclusion $\mathcal{U}_1 \hookrightarrow \mathcal{U}_2$ an injection $i_{\mathcal{U}_1,\mathcal{U}_2}\mathcal{A}(\mathcal{U}_1) \hookrightarrow \mathcal{A}(\mathcal{U}_2)$. Actually we would like some compatibility of composition of inclusion:

- For every open set $\mathcal{U}$, we require $i_{\mathcal{U},\mathcal{U}} = \mathrm{id}_{\mathcal{A}(\mathcal{U})}$,
- For any open sets $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_3$, we require $i_{\mathcal{U}_1,\mathcal{U}_3} = \imath_{\mathcal{U}_2,\mathcal{U}_3} \circ i_{\mathcal{U}_1,\mathcal{U}_2}$.

If we have $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subset O_3$ In mathematical terms, this means that we want to have a pre-cosheaf of C-$*$-algebras over M.

If $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, a function whose support is included in $\mathcal{U}$ can be expressed as a sum of a function with support included in $\mathcal{U}_1$ and a function included in $\mathcal{U}_2$. This translated into the following requirement: The algebra $\mathcal{A}(\mathcal{U})$ is generated as an algebra by $i_{\mathcal{U}_1,\mathcal{U}}(\mathcal{A}(\mathcal{U}_1)) + i_{\mathcal{U}_2,\mathcal{U}}(\mathcal{A}(\mathcal{U}_2))$.

We still want the rules of physics to be invariant under the change of space-time frame. For every element g of $\overline{\mathcal{P}}$ and every open set $\mathcal{U}$ in M, we want to have isomorphisms $\alpha_{g,\mathcal{U}} : \mathcal{A}(\mathcal{U}) \to \mathcal{A}(g\mathcal{U})$ with the following properties:

$$\alpha_{g',g\mathcal{U}} \circ \alpha_{g,\mathcal{U}} = \alpha_{g'g,\mathcal{U}} \quad \text{and} \quad \alpha_{g^{-1},\mathcal{U}} = \alpha^{-1}_{g,g^{-1}\mathcal{U}}.$$

This is the **equivariance axiom**.

To translate the **causality axiom** we need to consider $\mathcal{U}_1$ and $\mathcal{U}_2$ two space-like separated open subsets of M. We would like to say that the two algebras commute. This has a meaning only if we can compare these algebras. We can write:

$$[i_{\mathcal{U}_1,\mathcal{U}}(\mathcal{A}(\mathcal{U}_1)), i_{\mathcal{U}_2,\mathcal{U}}(\mathcal{A}(\mathcal{U}_2))] = \{0\},$$

where $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$.

The **time-slice axiom** can be rephrased like this: the evolution of the system follows some local rules. Therefore, we expect that the algebra $\mathcal{A}(\mathcal{U})$ to be the same as the algebra $\mathcal{A}(\widehat{\mathcal{U}})$ where $\widehat{\mathcal{U}}$ is the set of all points which are not space-like separated from $\mathcal{U}$: This means that the injection $i_{\mathcal{U},\widehat{\mathcal{U}}}$ is an isomorphism.

If we want to have the Hilbert space back, we want to *represent* the net of algebras $\mathcal{A}(\bullet)$ on an Hilbert space $\mathcal{H}$. This means the following: for each open set $\mathcal{U}$, we have a morphisms of algebras $\phi_{\mathcal{U}} : \mathcal{A}(\mathcal{U}) \to \mathcal{O}(\mathcal{H})$ compatible with the injections: if $\mathcal{U}' \subset \mathcal{U}$, we should have:

$$\phi_{\mathcal{U}'} = \phi_{\mathcal{U}} \circ i_{\mathcal{U}',\mathcal{U}}.$$

We should have a group morphism $\Psi : \overline{\mathcal{P}} \to U(\mathcal{H})$ (where $U(\mathcal{H})$ is the group of unitary transformation of $\mathcal{H}$) which intertwines the α's: We can think of $U(\mathcal{H})$ as acting on $\mathcal{O}(\mathcal{H})$, with this formalism we want to have for each $g \in \overline{\mathcal{P}}$ and each open set $\mathcal{U}$:

$$\Psi(g) \circ \phi_{\mathcal{U}} = \phi_{g\mathcal{U}} \circ \alpha_{g,\mathcal{U}}.$$

The **vacuum** and the **spectrum axiom** can phrased exactly like in the Wightmann setting.

Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany
E-mail address: louis-hadrien.robert@uni-hamburg.de