ZMP Hamburg

Wednesday, April 6, 2016 5:18 PM

AQFT

$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ — $*$-algebra of observables

$\mathcal{O} \subset \mathbb{M}$: region of spacetime (e.g. Minkowski)

pQFT

perturbative methods
Feynman graphs

$\boxed{\text{pAQFT}}$

Axioms:

1) Isotony $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$; $\mathfrak{A} := \lim\limits_{\mathcal{O} \subset M} \mathfrak{A}(\mathcal{O})$

2) Causality : if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike, then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$

3) Time-slice axiom : if $\mathcal{N}$ is a neigh. of a Cauchy surface Σ, then $\mathfrak{A}(\mathcal{N}) \cong \mathfrak{A}$

Physical input:

1) Spacetime : Smooth manifold M of dimension D with a smooth metric g of signature $(+, \underbrace{-, \ldots, -}_{D-1})$.

Assume : oriented, time-oriented, globally hyperbolic, i.e. $M = \Sigma \times \mathbb{R}$, where Σ is a Cauchy surface

E.g.: $M = \mathbb{M} \doteq (\mathbb{R}^4, \eta)$, where $\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

2) Space of field configurations $\mathcal{E} \doteq \Gamma(E \to M)$, E a vector bundle over M

E.g.: $\mathcal{E} = \mathcal{C}^\infty(M, \mathbb{R})$ for the scalar field

3) Observables : modeled by smooth functionals on $\mathcal{E}$.

Def: A derivative of a functional F at $\varphi \in \mathcal{E}$ in the direction of $h \in \mathcal{E}$:

$$\langle F^{(1)}(\varphi), h \rangle \doteq \lim_{t \to 0} \frac{1}{t} \left(F(\varphi + t h) - F(\varphi) \right)$$

F is continuously diff. if $F^{(1)} : \begin{array}{l} (\varphi, h) \longmapsto \langle F^{(1)}(\varphi), h \rangle \\ \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{C} \end{array}$ is cont. in the product topology

Def: A functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ is called local if it is of the form:

$$F(\varphi) = \int_M f(j^k(\varphi))\, d\mu_g \quad , \quad \varphi \in \mathcal{E}$$

finite jet of φ

volume form on M induced by g

Def: Define a product "$\cdot$" $(F \cdot G)(\varphi) \doteq F(\varphi)\, G(\varphi)$

Def.: Define $\mathfrak{F}$ (the space of multilocal functionals) as the algebraic completion of $\mathfrak{F}_{loc}$ wrt "$\cdot$".

Def: A functional is regular ($F \in \mathfrak{F}_{reg}$) if $F^{(n)}(\varphi)$ is smooth $\forall\ \varphi \in \mathcal{E}$, $n \in \mathbb{N}$

i.e. $F^{(n)}(\varphi) \in \Gamma_c((E^*)^{\boxtimes n} \to M^n)^{\mathbb{C}}$ (in general $F^{(n)}(\varphi) \in \Gamma'((E^*)^{\boxtimes n} \to M^n)^{\mathbb{C}}$)

where E^* is the dual bundle to E

Def: Support of a functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ is defined by:

$$\operatorname{supp} F \doteq \{x \in M \mid \forall O \text{ open neigh. of } x,\ \exists\ \varphi, \psi \in \mathcal{E} \text{ such that } \operatorname{supp}\psi \subset O \text{ and } F(\varphi+\psi) \neq F(\varphi)\}$$

↪ provides a notion of spacetime localization for functionals

4) Dynamics. Provided by a (local) 1-form on $\mathcal{E}$ denoted by S'.

Equation of motion: $S'(\varphi) = 0$ (i.e. solution space $\mathcal{E}_S$ is the zero locus of S')

I. Free theory

Assume that S_0' is a linear op. on $\mathcal{E}$, i.e. $S_0'(\varphi) = P\varphi$.
Assume P to be normally hyperbolic

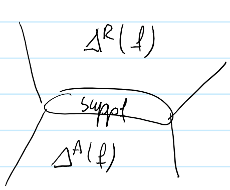
E.g.: for the free scalar field $P\varphi = -(\underbrace{\square}_{\text{wave op.}} + \underbrace{m^2}_{m^2 \in \mathbb{R}_+ \text{ mass}})\varphi$

Fact: There exist Δ^R, Δ^A retarded/advanced Green's functions for P, i.e.:

$$\Delta^{R/A}: \mathcal{E}_c^{*\mathbb{C}} \to \mathcal{E}^{\mathbb{C}}, \quad \text{where } \mathcal{E}_c^* \doteq \Gamma_c(E^* \to M)$$

i) $P \circ \Delta^{A/R} = id$, $\Delta^{R/A} \circ P|_{\mathcal{E}_c^{*\mathbb{C}}} = id$

ii) $\operatorname{supp}(\Delta^{R/A}(f)) \subset J^{\pm}(\operatorname{supp} f)$



Def: Define the commutator function (the causal propagator): $\Delta \doteq \Delta^R - \Delta^A$ (is antisymmetric)

Def: Define a Poisson bracket $\lfloor F, G \rfloor(\varphi) \doteq \langle F^{(1)}(\varphi), \Delta G^{(1)}(\varphi) \rangle$, $F, G \in \mathfrak{F}$

Rem: $\lfloor \cdot, \cdot \rfloor$ well def. for $F, G \in \mathfrak{F}_{loc}$, since $F^{(1)}(\varphi), G^{(1)}(\varphi) \in \mathcal{E}_c^{*\mathbb{C}}$, but $\mathfrak{F}$ not closed under $\lfloor \cdot, \cdot \rfloor$

One can consider a larger space $\mathfrak{F}_{\mu c}$ (microcausal functionals) such that $(\mathfrak{F}_{\mu c}, \lfloor \cdot, \cdot \rfloor)$ is a Poisson alg.
Technical tool: WF set.

Fact: We can split $\frac{i}{2}\Delta = \Delta_+ - \underbrace{H}_{\text{symmetric}}$, so that Δ_+ has a "nice" WF set (choice of Δ_+, H not unique)

Interpretation: if Δ_+ is a 2-point function of a pure state, we have a Kähler structure with $J \doteq 2\Delta_S^{-1} \circ H$

Def: Star product $F \star_H G(\varphi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \langle F^{(n)}(\varphi), \Delta_+^{\otimes n} G^{(n)}(\varphi) \rangle$, $F, G \in \mathcal{F}_{\mu c}$ and $\mathcal{F}_{\mu c}[[\hbar]]$ is closed under $\star_H$

$\Delta_+ = \frac{i}{2}\Delta + H$

Rem: Different choices of H related by:

$$F \star_{H'} G = \alpha_{H-H'}^{-1} (\alpha_{H-H'} F \star \alpha_{H-H'} G), \text{ where } H-H' \text{ is smooth and}$$

$$\alpha_H \doteq e^{\frac{\hbar}{2} D_H}, \quad D_H \doteq \langle H, \frac{\delta^2}{\delta\varphi^2} \rangle \quad \text{(gauge transformation)}$$

II. Interaction

On the example of scalar fields. For the moment restrict to $\mathcal{F}_{reg}$.

Def: The Feynman propagator corresponding to H is defined by

$$\Delta^F \doteq \frac{i}{2}(\Delta^R + \Delta^A) + H \quad \text{(is symmetric)}$$

Def: The time-ordered product $\cdot_T$ on $\mathcal{F}_{reg}$:

$$(F \cdot_T G)(\varphi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \langle F^{(n)}(\varphi), (\Delta^F)^{\otimes n} G^{(n)}(\varphi) \rangle$$

Prop.:

$$F \cdot_T G = \begin{cases} F \star_H G & \text{if supp } F \gtrsim \text{supp } G \\ G \star_H F & \text{if supp } G \gtrsim \text{supp } F \end{cases}$$

Hence "time-ordered"

Def: Formal S-matrix corresponding to the interaction λF:

$$\mathcal{S}(\lambda F) \doteq e_T^{\frac{i\lambda}{\hbar}F} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\lambda}{\hbar}\right)^n \underbrace{F \cdot_T \ldots \cdot_T F}_{\equiv \mathcal{T}_n(F, \ldots, F)}$$

n-fold time-ordered product

Problem: Let $F_1, \ldots, F_n \in \mathcal{F}_{loc}$. $\mathcal{T}_n$ is well defined only if $\text{supp } F_i \cap \text{supp } F_j = \emptyset \;\; \forall i,j$

$$\boxed{\text{Extension of } \mathcal{T}_n\text{'s to arbitrary local arguments} \equiv \text{RENORMALIZATION}}$$

Needed because typical interactions are local, e.g. $\int_{\mathbb{M}} f(x) \frac{\lambda}{4!} \varphi^4(x)\, d^4x$ in φ^4-theory on $\mathbb{M}$.

III. Feynman graphs

Prop.: $\mathcal{T}_n = m \circ e^{\hbar \sum_{i<j} D_F^{ij}}$, where $D_F^{ij} \doteq \langle \Delta^F, \frac{\delta^2}{\delta\varphi_i \delta\varphi_j} \rangle$

pointwise multiplication — derivative in i-th and j-th slot

$$\ldots \infty \; (\hbar D_F^{ij})^{l_{ij}}$$

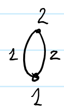
Fact: $e^{\hbar \sum_{i<j} D_F^{ij}} = \prod_{i<j} \sum_{l_{ij}=0}^{\infty} \frac{\left(\hbar D_F^{ij}\right)^{l_{ij}}}{l_{ij}!}$

Hence $\mathcal{T}_n = \sum_{\Gamma \in \mathcal{G}_n} \mathcal{T}_\Gamma$, $\mathcal{G}_n$ set of graphs with n vertices
l_{ij} is the number of lines connecting i & j
set $l_{ii} = 0$ $l_{ij} = l_{ji}$ for $i>j$

(*) $\mathcal{T}_\Gamma = \frac{1}{\mathrm{sym}(\Gamma)} m \circ \langle t_\Gamma, \delta_\Gamma \rangle$, where

$$\delta_\Gamma = \frac{\delta^{2|E(\Gamma)|}}{\prod_{i \in V(\Gamma)} \prod_{e: i \in \partial e} \delta\varphi_i(x_{e,i})}$$

$$t_\Gamma = \prod_{e \in E(\Gamma)} \hbar \Delta^F(x_{e,i}, i \in \partial e)$$

E.g.: $\Gamma =$  $t_\Gamma = \hbar^2 \Delta^F(x_{11}, x_{12}) \Delta^F(x_{21}, x_{22})$

Fact: δ_Γ applied to $F \in \mathcal{F}_{loc}^{\otimes n}$ gives a distribution with support on:

$$\mathrm{Diag}_\Gamma = \{x_{e,i} = x_{f,i}, i \in \partial e \cap \partial f, e, f \in E(\Gamma)\} \subset \mathbb{M}^{2|E(\Gamma)|}$$

To make sense of (*) we need to be able to define a pullback of t_Γ through:

$\rho_\Gamma : \mathrm{Diag}_\Gamma \to \mathbb{M}^{2|E(\Gamma)|}$ $\quad (\rho_\Gamma(z))_{e,v} = z_v$ if $v \in \partial e$.

In the first instance $\rho_\Gamma^* t_\Gamma$ is a distrib in $\mathcal{D}'(\mathrm{Diag}_\Gamma \setminus \mathrm{DIAG})$

$$\mathrm{DIAG} = \{z \in \mathrm{Diag}_\Gamma \mid \exists v, w \in V(\Gamma), v \neq w : z_v = z_w\}.$$

Hence: $\boxed{\text{Construction of } \mathcal{T}_\Gamma\text{'s} \equiv \text{extension problem for distributions } \rho_\Gamma^* t_\Gamma}$

More details can be found e.g. here: http://www.springer.com/de/book/9783319258997