An alternative to perturbative QFT: lattice discretization of the Sine-Gordon model

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1 Introduction

In this ZMP-lecture we want to explore alternatives to perturbative QFT. We have already seen that ordinary perturbative QFT has certain imperfections:

- Computing Feynman diagrams means having to deal with diverging integrals in need of regularization: this is ill understood at best and mathematically flawed at worst.
- the series presented by Feynman diagrams are often asymptotic only and need to be reinterpreted. A possible way out is the application of resurgence techniques as presented by Daniele Dorigoni in a previous ZMP-colloquium. Of course another perspective on this matter is that the existence of asymptotic series leads to the accessability of non-perturbative effects through these techniques, and as such are more of a feature than an imperfection.
- Despite the remark above, perturbative QFT is not a natural setting to study non-pertubative effects, even though we know that they exist in some QFT's. Examples of non-perturbative effects are the existence of instantons, solitons, confinement and Quark-gluon like plasmae.

Loosely speaking one big source of divergencies in QFT comes from the presence of small distance scales. The most naive way to avoid these small distances is to introduce a high-momentum cut-off: this, however, often breaks crucial symmetries of the theory, which can make doing actual computations rather difficult.

We will try here to follow a different scheme to avoid small distances, by studying the concept of *lattice regularizations*. A lattice regularization of a continuous theory in a general sense is a theory on a spacetime $\mathcal{M} \times \mathbb{L}$, with $\mathcal{M}$ a manifold and $\mathbb{L}$ a lattice with lattice spacing Δ, such that when Δ is sent to 0 we retrieve the continuous theory. What this exactly means can depend on the specific situation and interests: should we retrieve the lagrangian, the field equations, the symmetries, the expressions for observables, etc.?

There exist different approaches to lattice regularizations, related to the aims one might have. The most famous application of lattices in QFT might be lattice QCD, a way to make it possible to simulate QCD on computers. It played a big role in studying the quark-gluon plasma and is at present our best approach to QCD phenomenology.

Building a lattice model that contains enough of the qualities of its continuum version is nontrivial in itself. Moreover, one can take different stances in which of the models in the end truely is the "defining quantum theory": is it the mathematically flawed original QFT or the mathematically rigorous lattice theory that we have built to reproduce the QFT in the $\Delta \to 0$ limit? This is of course related to the question whether the lattice theory is even unique.

In this lecture, we would like to study the quantization of a specific classical field theory, the classical Sine-Gordon model, through the method of lattice regularization. We will see that *integrability* in particular will help us constrain the options we have in discretizing the classical theory.

2 Classical Sine-Gordon Model

The classical Sine-Gordon Model is a 2d-field theory and can be characterized as a hamiltonian system. Its hamiltonian is

$$H = \int_I dx \left(\frac{1}{2}\pi(x)^2 + \frac{1}{2}\left(\frac{d}{dx}\phi(x)\right)^2 + \frac{m^2}{\beta^2}(1 - \cos(\beta\phi(x))) \right) \tag{1}$$

with Poisson bracket $\{\pi(x), \phi(y)\} = \delta(x - y)$ and $\pi = \frac{d}{dt}\phi$. Also, $\pi(x) = \pi(x, 0)$, $\phi(x) = \phi(x, 0)$. The interval $I \subseteq \mathbb{R}$ can be many things, like $\mathbb{R}$, $[-L, L]$ and $[0, \infty)$. ϕ obeys the classical EOM:

$$\left(\partial_t^2 - \partial_x^2\right)\phi + \frac{m^2}{\beta}\sin(\beta\phi) = 0. \tag{2}$$

Using the time-evolution equation

$$\partial_t O(t) = \{H, O(t)\} \tag{3}$$

one can check that there are two conserved charges P, Q:

$$\begin{aligned} P &= -\int_I dx \pi(x) \frac{d}{dx}\phi(x) \\ Q &\sim \int_I dx \frac{d}{dx}\phi(x), \end{aligned} \tag{4}$$

where the last charge is topological.

2.1 Solutions of this EOM

Classically, there exist a few classes of solutions to this PDE, depending on the boundary conditions: there are soliton, antisoliton and breather solutions. For us, the existence of solitons is most interesting, nondispersive solutions. A single soliton solution is given by

$$\phi^{\pm}_{sol}(x, t) = \frac{4}{\beta}\arctan(e^{\pm m \frac{x - vt}{\sqrt{1 - v^2}} + \delta}) \tag{5}$$

with v, δ constants. The sign distinguishes between kink and anti-kink solutions (or soliton and antisoliton) and the v-dependence is due to a freedom in reference frame. However, there also exist multi-soliton solutions which can be interpreted as bound states. For example, the soliton-antisoliton solution usually called *breather* is given by

$$\phi_{breather}(x,t) = \frac{4}{\beta} \arctan \left(\frac{\sqrt{1-v^2} \sin(mvt)}{v \cosh\left(mx\sqrt{1-v^2}\right)} \right) \tag{6}$$

with free parameter $v \in (0,1)$ and can be interpreted as two particles scattering periodically in time. The general multiparticle solution can be found by Bäcklund transformations:

$$\phi(x,t) = \frac{4}{\beta} \arctan \frac{\mathrm{Im}(s)}{\mathrm{Re}(s)}, \tag{7}$$

with s being given by

$$s = \sum_{\mu_j=0,1} \exp\left(-\sum_{j=1}^{N} \mu)j \left[xm\cosh\theta_j + tm\sinh\theta_j - x_j - i\pi/2\epsilon_j\right] + 2\sum_{i<j} \mu_i\mu_j \log\frac{\tanh(\theta_i-\theta_j)}{2} \right). \tag{8}$$

For a state with k solitons, $N-k$ anti-solitons we set $\epsilon_i = 1$ for $i = 1, \cdots, k$ and $\epsilon_i = -1$ for $i = k+1, \cdots, N$. The parameters θ_j are free parameters (called *rapidities*) of the particles which have positions x_j. Apart from describing solitons and anti-solitons, this equation also describes bound states of solitons and anti-solitons, called *breathers*. They correspond to rapidity pairings $(\theta + i\vartheta, \theta - i\vartheta)$. These solutions display scattering behaviour that is a trademark of integrable models: multiparticle scattering can be broken down to subsequent 2-particle scatterings.

One would expect all the time-periodic solutions given above to become bound states of the quantum theory, using the Bohr-Sommerfeld quantization formula: for a one-parameter family of classical solutions ϕ_T indexed by the period T we have that, whenever

$$\int_0^T dt \int dx \pi(x,t) \partial_t \phi_T(x,t) = 2\pi n, \tag{9}$$

for some integer n, we expect there to be a corresponding energylevel in the quantum spectrum. More specifically, it predicts a set of energies

$$E_n = 2m/\gamma \sin(\gamma n/2), \tag{10}$$

with $\gamma = \beta^2/8$. We expect to need more than normal perturbative QFT to check this claim, since solitons are non-perturbative solutions.

2.2 How to quantize?

In perturbative QFT, we are interested in the lagrangian, since it is present in the path-integral. Classically the lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{\beta}\left(\cos(\beta\phi) - 1\right), \tag{11}$$

which for $\phi' = \beta\phi$ reads

$$\beta^2 \mathcal{L} = \frac{1}{2}\partial_\mu \phi' \partial^\mu \phi' + m^2 \left(\cos\phi' - 1\right). \tag{12}$$

In the action we have the quantity $\mathcal{L}/\hbar$ standing, so

$$\mathcal{L} = \frac{1}{\beta^2 \hbar}\left(\frac{1}{2}\partial_\mu \phi' \partial^\mu \phi' + m^2 \left(\cos\phi' - 1\right)\right). \tag{13}$$

The classical limit coincides with $\beta \to 0$, so we do not need $\hbar$ separately and set it to 1. Following the normal perturbative QFT folkore, we study Feynman diagrams and run into UV-divergencies for diagrams that have loops that start and finish at the same vertex. They can be removed by normal ordering the interaction term in the lagrangian. However, this normal ordering is ambiguous, since we are not sure yet which vacuum to normal-order against. Moreover, variational methods can be used to predict that the theory is unbounded from below if $\beta^2 > 8\pi$, rendering the theory inconsistent. This cannot be checked by perturbation theory, requiring new methods. Also, the normal ordering procedure we are treating here seems to break a lot of the symmetries present at the classical level, implying it might not be the best way to quantize the theory.

2.3 Lax representation and Classical Integrability

We will later use the existing classical integrable structure of the sine-gordon model as a guide when discretizing. We will therefore first summarize the most important aspects of this structure: there exists a Lax representation (or sometimes called *zero-curvature condition*) for the sine-gordon equation:

$$[\partial_t - U(x,t|\lambda), \partial_x - V(x,t|\lambda)] = 0, \tag{14}$$

is equivalent to

$$(\partial_t^2 - \partial_x^2)\phi + \frac{8m^2}{\beta}\sin(2\beta\phi) = 0, \tag{15}$$

where $\lambda \in \mathbb{C}$ and

$$\begin{aligned} U(x,t|\lambda) &= i\begin{pmatrix} \partial_x \phi(x)\beta/2 & m\left(\lambda e^{-i\beta\phi(x)} + \lambda^{-1} e^{i\beta\phi(x)}\right) \\ m\left(\lambda e^{i\beta\phi(x)} + \lambda^{-1} e^{-i\beta\phi(x)}\right) & -\partial_x \phi(x)\beta/2 \end{pmatrix} \\ V(x,t|\lambda) &= i\begin{pmatrix} \pi(x)\beta/2 & -m\left(\lambda e^{-i\beta\phi(x)} - \lambda^{-1} e^{i\beta\phi(x)}\right) \\ -m\left(\lambda e^{i\beta\phi(x)} - \lambda^{-1} e^{-i\beta\phi(x)}\right) & -\pi(x)\beta/2 \end{pmatrix}. \end{aligned} \tag{16}$$

This is a consistency condition for the 2-2 matrix equations

$$\partial_t \psi = U(x,t|\lambda)\psi \tag{17}$$

$$\partial_x \psi = V(x,t|\lambda)\psi. \tag{18}$$

the *transition matrix* $T(x,y|\lambda)$ satisfies

$$\begin{aligned} (\partial_x - V(x,t|\lambda))T(x,y|\lambda) &= 0 \\ T(y,y|\lambda) &= \mathbb{I} \\ \psi(x,t) &= T(x,y|\lambda)\psi(y,t) \end{aligned} \tag{19}$$

suppressing t-dependence. We can now see the integrable structure of the model, since the potential V satisfies

$$\{V(x,t|\lambda) \overset{\otimes}{,} V(y,t|\mu)\} = \delta(x-y)\left[r(\lambda/\mu), V(x,t|\lambda)\otimes \mathbb{I} + \mathbb{I}\otimes V(x,t|\mu)\right], \tag{20}$$

with the *r-matrix* given by

$$r(\lambda) = \pi\beta^2\left(\lambda-\lambda^{-1}\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda+\lambda^{-1} & -2 & 0 \\ 0 & -2 & \lambda+\lambda^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

The expression $\{M \overset{\otimes}{,} N\}$ for 2-2 matrices $M, N \in \mathrm{Mat}(2,\mathbb{C})$ is defined as

$$\{M \overset{\otimes}{,} N\} = \begin{pmatrix} \{M_{11},N_{11}\} & \{M_{11},N_{12}\} & \{M_{12},N_{11}\} & \{M_{12},N_{12}\} \\ \{M_{11},N_{21}\} & \{M_{11},N_{22}\} & \{M_{12},N_{21}\} & \{M_{12},N_{22}\} \\ \{M_{21},N_{11}\} & \{M_{21},N_{12}\} & \{M_{22},N_{11}\} & \{M_{22},N_{12}\} \\ \{M_{21},N_{21}\} & \{M_{21},N_{22}\} & \{M_{22},N_{21}\} & \{M_{22},N_{22}\} \end{pmatrix} \tag{22}$$

This implies

$$\{T(x,y|\lambda) \overset{\otimes}{,} T(x,y|\mu)\} = \left[T(x,y|\lambda)\otimes T(x,y|\mu), r(\lambda/\mu)\right], \tag{23}$$

which implies that the *transfer matrix*

$$\tau(\lambda) = \mathrm{Tr}_{\mathbb{C}^2}\left(T(L,0|\lambda)\right) \tag{24}$$

satisfies

$$\{\tau(\lambda), \tau(\mu)\} = 0. \tag{25}$$

Since we can find the hamiltonian of the Sine-Gordon model as a linear combination of coefficients c_n in

$$\log\left(\tau(\lambda)\right) = \sum_{n=-\infty}^{\infty} c_n\lambda^n, \tag{26}$$

we see that τ is time-independent and thus generates an infinite number of integrals of motion: there exists an $M \in \mathbb{N}$ such that

$$H_{\text{SG}} = \sum_{n=-M}^{M} a_n c_n \text{ with } a_n \in \mathbb{C}. \tag{27}$$

Note however, that if we are only interested in the spectrum we do not need anything else but the spectrum of T: it commutes with the hamiltonian, so they share a common basis of eigenvectors. Since eigenvectors are parametrized by momentum, we can use the relativistic energy equation to find the corresponding eigenvalue.

3 Lattice Sine-Gordon model: classically

Take $I = [0, L]$, $\Delta > 0$ and define

$$x_n = \Delta n \text{ for } n = 0, 1, \cdots, N \tag{28}$$

such that the total number of sites on our lattice is $N = L/\Delta$. Define

$$\begin{aligned} \phi_n &:= \phi(x_n) \approx \frac{1}{\Delta}\int_{x_{n-1}}^{x_n} \phi(z)dz \\ \pi_n &:= \Delta\pi(x_n) \approx \int_{x_{n-1}}^{x_n} \pi(z)dz \end{aligned} \tag{29}$$

We return to the auxiliary problem

$$\partial_t\psi = U(x,t|\lambda)\psi \tag{30}$$

$$\partial_x\psi = V(x,t|\lambda)\psi. \tag{31}$$

Consider this equation at x_n and discretize:

$$\begin{aligned} \partial_t\psi(x_n,t) &= U(x_n,t|\lambda)\psi(x_n,t) \\ \frac{\psi(x_{n+1},t)-\psi(x_n,t)}{\Delta}) = V(x_n,t|\lambda)\psi(x_n,t) \Rightarrow \psi(x_{n+1},t) &= \left(\mathbb{I}+\Delta V(x_n,t|\lambda)\right)\psi(x_n,t). \end{aligned} \tag{32}$$

Additionally, we can discretize the Poisson bracket:

$$\{\pi_n, \phi_m\} = \delta_{nm}. \tag{33}$$

We search for an operator $L(n|\lambda)$ such that

$$\begin{aligned} L(n|\lambda) &= \mathbb{I} + \Delta V(x_n,t|\lambda) + \mathcal{O}(\Delta^2) \\ &= \begin{pmatrix} 1+\frac{i\beta\Delta}{2}\pi_n & -im\Delta\left(\lambda e^{-i\beta\phi_n} - \lambda^{-1}e^{i\beta\phi_n}\right) \\ -im\Delta\left(\lambda e^{i\beta\phi_n} - \lambda^{-1}e^{-i\beta\phi_n}\right) & 1-\frac{i\beta\Delta}{2}\pi_n \end{pmatrix} + \mathcal{O}(\Delta^2) \end{aligned} \tag{34}$$

We define L to be

$$L(n|\lambda) = \begin{pmatrix} u_n + \kappa_n^2 v_n u_n v_n & -i\kappa_n\left(\lambda v_n - \lambda^{-1}v_n^{-1}\right) \\ -i\kappa_n\left(\lambda v_n^{-1} - \lambda^{-1}v_n\right) & u_n^{-1} + \kappa_n^2 v_n^{-1}u_n^{-1}v_n^{-1} \end{pmatrix}, \tag{35}$$

with

$$\begin{aligned} u_n &:= e^{i\frac{\beta}{2}\pi_n} \\ v_n &:= e^{-i\beta\phi_n} \end{aligned} \tag{36}$$

and κ_n being a scale parameter. In the continuum limit $\Delta \to 0$, $\kappa_n/\Delta \to m$. This defines a classically integrable lattice model, through the usual steps: we find that L satisfies

$$\{L(n|\lambda) \overset{\otimes}{,} L(m|\mu)\} = \delta_{mn}\left[L(n|\lambda)\otimes L(m|\mu), r(\lambda/\mu)\right], \tag{37}$$

with the exact same r-matrix as before! Now define a transition matrix

$$T(n,m|\lambda) = L(n|\lambda)\cdots L(1|\lambda)L(m|\lambda) \text{ for } n \geq m. \tag{38}$$

From the relation for L we find that

$$\{T(n,m|\lambda) \overset{\otimes}{,} T(n,m|\mu)\} = [T(n,m|\lambda) \otimes T(n,m|\mu), r(\lambda/\mu)]\,. \tag{39}$$

Again defining the monodromy matrix

$$\tau(\lambda) = \mathrm{Tr}_{\mathbb{C}^2}\left(T(N-1,0|\lambda)\right) \tag{40}$$

we find that $\{\tau(\lambda),\tau(\mu)\} = 0$. So the coefficients c_n in

$$\log\left(\tau(\lambda)\right) = \sum_{n=-\infty}^{\infty} c_n\lambda^n, \tag{41}$$

are integrals of motion for a dynamical model with as a hamiltonian H a linear combination of the c_n. In particular, there exists a linear combination

$$H \sim \sum_{n=-M}^{M} a_n c_n \text{ with } a_n \in \mathbb{C} \text{ and } M \in \mathbb{N}_0 \tag{42}$$

and find that

$$\lim_{\Delta\to 0} H = H_{\mathrm{CSG}}. \tag{43}$$

In this sense the model we constructed can really be regarded as an honest discretization of the CSG-model.

Another approach that is easier for quantization is to define a new parametrization

$$\begin{aligned} f_{2n,0} &= e^{-2i\beta\phi_n} \\ f_{2n-1,1} &= e^{i\beta/2(\pi_n+\pi_{n-1}-2\phi_n-2\phi_{n-1})} \end{aligned} \tag{44}$$

and imagine these variables as the starting condition for a dynamical theory on a two dimensional rhombic lattice defined by

$$\mathcal{L} = \{(\nu,\tau) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}\,|\,\nu+\tau \in 2\mathbb{Z}\}. \tag{45}$$

One can then propagate these starting conditions to the entire lattice using an evolution equation

$$f_{\nu,\tau+1} = U^{-1} f_{\nu,\tau-1} U, \tag{46}$$

where U is some operator built up from u, v. It is slightly harder to see the continuum limit from this new perspective, where also the time-direction has been discretized, but the evolution operator can be easily quantized after quantizing the operators u, v. See more in [1].

4 Lattice Sine-Gordon model: quantum

Now we can quantize the lattice model we just constructed, by promoting the functions π_n, ϕ_n to operators and their poisson bracket to a commutator:

$$\{\pi_n, \phi_m\} \to \frac{1}{i\hbar}[\hat{\pi}_n, \hat{\phi}_m] \tag{47}$$

$$[[\hat{\pi}_n, \hat{\phi}_m] = i\delta_{nm} \text{ with } \hbar = 1. \tag{48}$$

However, u, v behave better than π, ϕ, so instead of canonically quantizing π, ϕ and deriving the relations for u, v, we directly treat u, v as the fundamental variables, quantize those and search for representations of those operators and their algebra. They satisfy Weyl commutation relations:

$$\hat{u}_n \hat{v}_m = q^{\delta_{nm}} \hat{v}_m \hat{u}_n \text{ with } q = e^{-i\pi\beta^2}. \tag{49}$$

It seems obvious that special things could happen for those cases in which $q^l = 1$ for some integer l. These root of unity cases exist whenever $\gamma = \pi k$ with $k \in \mathbb{Q}$. For these cases the operators u, v will have discrete spectrum and the problem is most approachable. In any case, we can go through the same setup as in the classical case to see that we did in fact preserve integrability:

Using the commutation relations for u, v we can check that the quantized Lax matrix satisfies

$$R(\lambda/\mu)\left(\hat{L}(\lambda) \otimes I\right)\left(I \otimes \hat{L}(\mu)\right) = \left(I \otimes \hat{L}(\mu)\right)\left(\hat{L}(\lambda) \otimes I\right) R(\lambda/\mu) + \mathcal{O}(\kappa^2), \tag{50}$$

where the r-matrix R is now given by

$$R(\lambda) = \begin{pmatrix} q\lambda - q^{-1}\lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda - \lambda^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & \lambda - \lambda^{-1} & 0 \\ 0 & 0 & 0 & q\lambda - q^{-1}\lambda^{-1} \end{pmatrix}. \tag{51}$$

This property transfers to the transfer matrix

$$\hat{T}(\lambda) = \hat{L}_{N-1}(\lambda)\hat{L}_{N-2}(\lambda)\cdots\hat{L}_0(\lambda), \tag{52}$$

which in turn is used in the definition of the τ-operator:

$$\hat{\tau}(\lambda) = \mathrm{tr}_{\mathbb{C}^2}(\hat{T}(\lambda)). \tag{53}$$

By the commutativity of the $\hat{L}$ we can prove that

$$[\hat{\tau}(\lambda), \hat{\tau}(\mu)] = 0. \tag{54}$$

Ideally, we would like to find a hamiltonian as a linear combination of the Laurent coefficients of $\log \tau(\lambda)$, but this is a difficult problem that we won't treat here. However, the discrete time evolution operator U we have introduced before can be easily quantized and it can be shown that τ and U commute. This implies that again, only the spectrum of $\tau(\lambda)$ is enough to produce the spectrum of the quantum theory that we have just defined.

4.1 Spectrum

There are different ways of deriving equations for the spectrum. Historically, Algebraic Bethe Ansatz was the first available method, but might lack a certain rigour. A more modern approach is through Baxter's TQ-construction. The resulting equations can then be analyzed in a variety of different ways, including the Truncated Conformal Space approach, Thermodynamic Bethe Ansatz and the Non-Linear Integral Equations method introduced by Destri and De Vega. We will not have time to dive into all the intricate details of this analysis and instead only present the results. One can find much of this discussion in the work done by:

- Faddeev, Sklyanin, Takhtajan [2]
- Faddeev, Volkov [1]
- Niccoli, Teschner [3]
- Bajnok, Šamaj [4]
- Destri, De Vega [5, 6]
- Feverati, Ravanini, Takacs [7]

However, this list is far from being exhaustive and the interested reader is encouraged to spend a blissful afternoon going through the extensive literature on the topic.

All these approaches first derive a set of equations for the parameter λ, called *Bethe equations*:for a given $M \in \mathbb{N}$ and each $1 \leq j \leq M$ we have

$$-1 = \frac{\sinh \gamma/\pi(\lambda_j + i\pi/2 + \Theta) \sinh \gamma/\pi(\lambda_j + i\pi/2 - \Theta)}{\sinh \gamma/\pi(\lambda_j - i\pi/2 + \Theta) \sinh \gamma/\pi(\lambda_j - i\pi/2 - \Theta)} \prod_{k=1}^{M} \frac{\sinh \gamma/\pi(\lambda_j - \lambda_k + \pi i)}{\sinh \gamma/\pi(\lambda_j - \lambda_k - \pi i)}. \tag{55}$$

Here $\gamma = \beta^2/8$ and Θ is the reincarnation of the scaling function κ_n. The continuum limit for finite volume L now has become sending $\Theta \to \infty$ as we send $\Delta \to 0$ such that $\Delta e^{\Theta} \to 4/m$. This can be done rigorously, because after deriving the matching NLIE equations all the dependence on Δ sits in one source term for which the limit exists unambiguously.

We can analyze these equations for different values of γ and find the following: if $0 < \gamma < \pi/2$ the interaction between particles is repulsive and the spectrum consists of

- multi-solitons
- multi-antisolitons

The point $\gamma = \pi/2$ is special and corresponds to a CFT with a marginal deformation. For $\gamma > \pi/2$ the interaction of particles is attractive and we find extra particles in the spectrum: apart from the solitons and antisolitons, there start to appear socalled breathers. For increasing γ, the nth breather appears at $\gamma = \frac{n\pi}{n+1}$.

The found particle content matches the predictions from semiclassical analysis.

5 Conclusion

We have explored a method that allows us to quantize a classical field theory through the intermediate step of discretizing. In particular we have seen

- the role of integrability as a guide, restricting us in how to discretize/quantize
- that the resulting spectrum contains the particles that we expected from semi-classical analyses
- that the role of regulator was played by the lattice spacing Δ: connecting the lattice to the continuum through a specific limiting procedure.

Some unexplored interesting aspects of the Sine-gordon model: its relation to

- conformal field theory.
- the XXZ spin chain.
- the Massive Thirring model (see for example [8, 9]
- the Hirota equation (see [1]).

Some interesting observations: we where debating lattice integrable QFT or CFT perturbation theory. Quantum SG-model can be seen as a perturbation of a CFT:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{\beta^2} :\cos(\beta\phi(x)): \tag{56}$$

where the dots indicate some normal ordering. The potential term is marginal for $\beta^2 = 8\pi$.

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