Lóránt Szegedy
19.10.2017
ZMP Seminar

Zhu's algebra $A(V)$ & the C_2 algebra

I Motivation & Definitions [GG, Z]

$\boxed{V:\ \mathrm{VOA}}$ $\quad Y(a,z)=\sum_{n\in\mathbb{Z}} a(n) z^{-n-1}$

- Want to understand VOA modules to build full CFT.
- assign associative alg to VOA to study reps: as for grps & Lie-alg's.

Need more general notion of VOA modules:

$M=\bigoplus_{n\in\mathbb{N}} M_{(n)}$: $a(k)M_{(n)}\subset M_{(n+\deg a-k-1)}$

[If $M=\bigoplus M_\alpha$ s.t. $L_0|_{M_\alpha}=\alpha\, \mathrm{id}_{M_\alpha}$ $\Rightarrow$ $S:=\{s\in\mathbb{C}/\ M_s\neq 0\ \&\ M_{s-n}=0\ \forall n>0\}$

condition? $\quad \Rightarrow M_{(n)}:=\bigoplus_{s\in S} M_{s+n}$.]

Def $M':=\bigoplus_{n\in\mathbb{N}} M^*_{(n)}$: graded dual $\quad (f,u):=f(u)\quad f\in M',\ u\in M$.

contragredient module $\quad (Y_{M'}(a,z)f,u)=(f,Y_M(e^{zL_1}(-z^{-2})^{L_0}a,z^{-1})u)$

$\longrightarrow$ Look at 3-pt conf. block: $\quad f\in M'_{(0)}\quad u\in M_{(0)}\quad v\in V:\quad w\in\mathbb{P}^1\setminus\{0,\infty\}$

Def $\eta(v):=\langle f(\infty)\ v(0)\ u(w)\rangle$

$\underset{\text{(conf. tf.)}}{\overset{x\mapsto x-w}{=}}\langle f(\infty)\ v(-w)\,u(0)\rangle$

$=(f,Y_M(v,-w)u)$.

Idea: $*$ $v\in V$ for which $\eta(v)\neq 0$ know something about M.

$*$ can write down such v's easily:

$\longrightarrow$ Look at 4-pt conf. block: $\quad a,b\in V$, a homog. $z\in\mathbb{P}^1\setminus\{0,w,\infty\}$

$\langle f(\infty)\ a(z)\ b(0)\ u(w)\rangle=\langle f(\infty)\ a(z-w)b(-w)u(0)\rangle=(f,Y_M(a,z-w)Y_M(b,-w)u(0))$

$=(f,Y_M(Y(a,z)b,-w)u)$.

(single valued fn.)

Claim: $\uparrow$ has poles in z for $\quad z=w$ ord $\leq \deg a$; $z=\infty$, $z=0$ ord $\leq 2-\deg a$?!

Γ Switch Y_M's & expand $Y_M(a,z-w)$, use $a(n)M_{(0)}\subset M_{(\deg a-n-1)}$

$\Rightarrow \frac{1}{2\pi i}\oint_\gamma (f,Y_M(Y(a,z)b,-w)u)\cdot z^{-2-m}\cdot(z-w)^{\deg a}\,dz=0$

by "pulling" γ over to γ': Integrand has no pole.

$\parallel$

$(f,Y_M(\mathrm{Res}_z[Y(a,z)b\,z^{-2-m}(z-w)^{\deg a}],-w)u)$

Def $v_{a,b,w,m}$ $\boxed{1}$ $\quad \Rightarrow\ \eta(v_{a,b,w,m})$. $\quad$ [Res_z: Coeff of z^{-1}].

Def Let $v_{a,b,w,m} := \mathrm{Res}_z\left(Y(a,z)\, z^{-2-m}(z-w)^{\deg a}\, b\right) \in V$.

$\Rightarrow$ We just showed that $\boxed{\eta_w\left(v_{a,b,w,m}\right) = 0}$.

Def $O_w(V) := \mathrm{span}\{ v_{a,b,w,0} \mid a,b \in V \text{ a homog.}\}$

Lem 1 [Z, Lem 2.1.2] $v_{a,b,w,m} \in O_w(V)$ $\forall m \geq 0$.
$(w=-1)$ (adjust pf for arb w)

Def $A_w(V) := V/O_w(V)$; $A(V) := A_{-1}(V)$ Zhu's algebra

Rem $\forall\, w,w' \in \mathbb{P}^1 \setminus \{0,\infty\}$ $A_w(V) \simeq A_{w'}(V)$.

Def $a,b \in V$, a homog.: $a *_w b := \mathrm{Res}_z\left[Y(a,z)\, z^{-1}(z-w)^{\deg a}\, b\right]$

$V \otimes V \to V$.

Thm [Z, Thm 2.1.1] $A(V)$ is an associative algebra w/ product $* = *_{-1}$.
$[1]$ is the unit and $[\omega]$ is a central element.

$\to$ this algebra will tell us about modules of V;
but first let us define another algebra...

Def

By simple computation: $v_{a,b,w,0} = \sum_{\ell=0}^{\deg a} \binom{\deg a}{\ell} (-w)^\ell\, a(\ell-2) b$;

$a *_w b = \sum_{\ell=0}^{\deg a} \binom{\deg a}{\ell} (-w)^\ell\, a(\ell-1) b$.

Taking the $\boxed{\text{limit } w \to 0}$: only 1 term survive.

Def $v_{a,b,0} := a(-2)b$ $\quad a *_0 b := a(-1)b \quad C_2(V) := \mathrm{span}\{a(-2)b\}$

Thm [Z, Sec. 4.4] $V/C_2(V)$ is a commutative associative Poisson algebra w/ P.b. $\{a,b\} := a(0)b$.

$\to$ one can see $A(V)$ as a deformation of $V/C_2(V)$.

Def The VOA V is called C_2-cofinite if $V/C_2(V)$ is finite dimensional.

Rem
- We will use this condition next time to prove things on chars.
- Every C_2-cofinite VOA has finitely many simple modules.

Prop 1 $\dim A(V) \leq \dim V/C_2(V)$.

(Idea) One can define a surjection for every $a,b \in V$ homog.:

$$\mathcal{O}(V) := \mathcal{O}_{-1}(V) \twoheadrightarrow C_2(V)$$

$$V_{a,b,-1} = \underbrace{a(-2)b}_{\in V_{\deg a+\deg b-1}} + \sum_{e=1}^{\deg a} \underbrace{a(e-2)b}_{\in V_{\deg a+\deg b-1-e}} \longmapsto a(-2)b.$$

II Modules of V and of A(V)

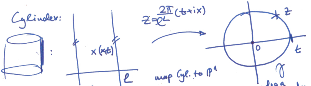
Def Let $a \in V$ be homogeneous. $o(a) := a(\deg a - 1) \in \operatorname{End} V$
$= \operatorname{Res}_z\left(Y(a,z)z^{\deg a-1}\right)$

Rem * for $M \in V\text{-Mod}$ $\quad o(a)\,M(\overset{m}{0}) \subset M(\overset{m}{0})$ $\forall m \geq 0$
"zero mode"

* Assume that a is primary i.e. $L_n a = 0 \;\forall n > 0$, $b \in V$.
$o(a)$ is the conserved charge of the conserved current $a(x,t)$.

Field a inserted at (x,t) on a cylinder.

Cylinder:



map Cyl. to $\mathbb{P}^1$ $\qquad$ up to factors of $\frac{2\pi}{\ell}$

$$G(a)b = \int_0^{\ell} a(x,t)b\,dx \;\overset{\downarrow}{=}\; \oint_\gamma a(z)\left(\frac{dz}{dx}\right)^{\deg a}\frac{dx}{dz}\,dz \;\overset{\downarrow}{\sim}\; \frac{1}{2\pi i}\oint a(z)b\,z^{\deg a-1}\,dz$$

$$= \operatorname{Res}_z\left[Y(a,z)z^{\deg a-1}\right]b = a(\deg a-1)b = o(a)b.$$

Thm 1 [Z, Thm 2.1.2] Let $M \in V\text{-Mod}$. Then $M(0) \in A(V)\text{-Mod}$
and $[a] \in A(V)$ acts as $o(a)$. $\quad o(a)o(b)|_{M(0)} = o(a*b)|_{M(0)}$

[* Showing associativity is a straightforward, but slightly long calc.
* Well-definedness: Will use VOA axiom $Y(L_{-1}a,z) = \frac{d}{dz}Y(a,z)$
and that $\operatorname{Res}_z\left[\left(\frac{d}{dz}f\right)g\right] = \operatorname{Res}_z\left[-f\left(\frac{d}{dz}g\right)\right]$.

$$\to \operatorname{Res}_z\left[Y(a,z)\,z^{-2}(z+1)^{\deg a}\,b\right] = \left((L_{-1}a) + \deg a\cdot a\right)*b$$

$$\to o\Bigg(\qquad\qquad \Bigg)\Bigg|_{M(0)} o\Bigg(\qquad\qquad \Bigg)\Bigg|_{M(0)}$$

$\overset{\text{assoc.}}{\Downarrow}$

$$o\left(L_{-1}a + \deg a\cdot a\right)o(b)\big|_{M_0}$$

this is 0 by a direct calculation.]

Thm [Z, Thm 2.2.1 & 2.2.2]

1) Let $W \in A(V)$-Mod. Then there exist $M \in V$-Mod s.t. $M[0] = W$ as $A(V)$-modules.

2) $\boxed{\text{Isomorphism classes of simple } V\text{-modules are in bijection w/ iso. classes of simple } A(V)\text{-modules.}}$

Rem [DLM]

* There are functors $V\text{-Mod} \underset{L}{\overset{\Omega}{\rightleftarrows}} A(V)\text{-Mod}$ s.t. $\Omega \circ L \simeq \mathrm{id}_{A(V)\text{-Mod}}$.
* If V is C_2-cofinite, then there are finitely many iso. classes of simple V-modules. [Follows from Rep 1]
* [Z, Thm 2.2.3] If V is rational then $L \circ \Omega = \mathrm{id}_{V\text{-Mod}}$ [DLM] and $A(V)$ is semi simple.

V is rational if every V-module is $\oplus$ of simples.

III Example: Lee–Yang model [G]

$\mathrm{Vir}: [L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0} \frac{(n+1)n(n-1)}{12} C$ ↑ central.

Let $c \in \mathbb{C}$ and let M_c be the Vir Verma module where $L_n|0\rangle$ for $n \geq 0$, C acts as c.

Let V_c denote its simple quotient.

V_c has the str. of a VOA w/ $\omega = L_{-2}|0\rangle$. (here we mean the equiv cl., but do not write it.)

The unique max ideal J_c of M_c always contains $L_{-1}|0\rangle$

$$L_n L_{-1}|0\rangle = (n+1) L_{-1} L_n |0\rangle + \delta_{n,1} \frac{(n+1)n(n-1)}{12} C|0\rangle = 0.$$

For the Lee–Yang model $\boxed{c = -\frac{22}{5}}$. There J_c is generated by $L_{-1}|0\rangle$ and $L_{-4}|0\rangle - \frac{5}{3} L_{-2}^2 |0\rangle = \mathcal{N}$

From Lem 1: $\mathrm{Res}_z \left(Y(\omega, z) \frac{(z+1)^2}{z^{2+m}} |0\rangle \right) \in O(V_c) \quad \forall m \geq 0$

$\parallel \;=\; V_{\omega, \omega_1 - 1, m}$

$$(L_{-3-m} + 2L_{-2-m} + L_{-1-m})|0\rangle$$

$\Rightarrow$ In $A(V_c) = V_c / O(V_c)$ we can express $[L_{-n}|0\rangle]$ as multiples of $[L_{-2}|0\rangle]$ $\forall n \geq 3 \Rightarrow A(V_c) = \mathrm{span}\{ L_{-2}^k |0\rangle \mid k \in \mathbb{N} \}$

$\Rightarrow$ Using that $\mathcal{N} \in J_c$ we have $[L_{-4}|0\rangle] = [\frac{5}{3} L_{-2}^2 |0\rangle]$,

so combining this with the above we have

$A(V_c) = \mathrm{span}\{ [|0\rangle], [\omega] = [L_{-2}|0\rangle] \}$ i.e. a polynomial alg mod some relation.

$\Rightarrow$ Calculate the relation $[\omega] * [\omega] = ?$

In $A(V_c)$: $\omega * \omega = \mathrm{Res}_z \left[Y(\omega, z) \frac{(z+1)^2}{z} \omega \right] = (L_{-2} + 2L_{-1} + L_0) L_{-2} |0\rangle$

$$\overset{\mathrm{Vir}}{=} (L_{-2}^2 + 2L_{-3} + 2L_{-2}L_{-1} + 2L_{-2})|0\rangle$$

$$\overset{\mathcal{N}}{=} \left(\tfrac{5}{3} L_{-4} + 2L_{-3} + 2L_{-2}\right)|0\rangle \underset{\text{Lem 1}}{=} \left(\tfrac{4}{5} L_{-3} + \tfrac{7}{5} L_{-2}\right)|0\rangle$$

$$= -\tfrac{1}{5} L_{-2}|0\rangle = -\tfrac{1}{5}\omega.$$

$\Rightarrow A(V_{-\frac{22}{5}}) = \mathbb{C}[\omega] / \omega(\omega + \frac{1}{5}) \Rightarrow V_{-\frac{22}{5}}$ has 2 irreps $\to L_0|_{M(0)} = 0 \to M = V_{-\frac{22}{5}}$

$\to L_0|_{M'(0)} = -\frac{1}{5}$

Ising: $A(V_{1/2}) \cong \mathbb{C}[\omega] / \omega(\omega - \frac{1}{2})(\omega - \frac{1}{16})$

[DLM] Dong, Li, Mason: q-alg/9509005
[G] Gaberdiel: hep-th/0111260
[GG] Gaberdiel, Gannon: 0811.3892
[Z] Zhu: J. Amer. Math. Soc. 1996 (9), 237–302.