ZMP talk: C_2-cofiniteness and torus 1-pt CB's

Dienstag, 31. Oktober 2017 20:57

References:

[Z]: Zhu - Modular invariance of characters of VOA's

[GK]: Gaberdiel, Keller - Modular differential eq's and null vectors

[AN]: Arike, Nagatomo - Some remarks on pseudo-trace fctn's for orbifold models assoc. with symplectic fermions

Main result of the talk:

Zhu: Let V be a VOA, s.t.

remember Lorenz's talk:

(i) V is C_2-cofinite ← C_2 cofinite $\Leftrightarrow$ $V/C_2(V)$ f.d.

↪ in part.: finitely many irrep's

(ii) every module is a direct sum of irrep's (semisimple)

$\Rightarrow$ (1) $Ch(M_i)(q) := tr_{M_i}\, q^{L_0 - \frac{c}{24}} = \sum_{n=0}^{\infty} \dim(M_{i,n})\, q^{n-\frac{c}{24}}$ converge for $|q|<1$

These are the "chiral halfs" of the 1-pt correlators on the torus from Lorenz' talk. Without assumptions it is not clear why these should be holomorphic fctn's.

(2) The space spaned by the holomorphic fctn's $Ch(M_i)(q = e^{2\pi i \tau}) \in Hol(\mathbb{H})$ carries an $SL(2,\mathbb{Z})$-action.

Remarks:

! · Semisimplicity: 1-pt-fctn's span space of 1-pt CB's

· short explanation: interpret them as 1-pt conformal blocks on torus!

· It is furthermore true that this action is compatible with the canonical $SL(2,\mathbb{Z})$-action on the MTC of V-modules.

§1. 1-pt functions on the torus

In order to prove the above result, it is helpful to look at q-traces with one insertion:

(1) $tr|_M\, Y_M(a,z)\, q^{L_0}$

this corresponds to the geometric process of identifying w and qw in $\mathbb{C}\setminus\{0\}$
$\to$ torus: $\mathbb{C}\setminus\{0\} / \{w \sim wq^n\}$

Remember: module: $\left(M = \bigoplus_{n\in\mathbb{Z}} M_n,\ Y_M(-,z)\right)$, s.t. (fin. dim.; bounded from below)

$a_n|_{M_k}: M_k \longrightarrow M_{k+\deg a-1-n}$

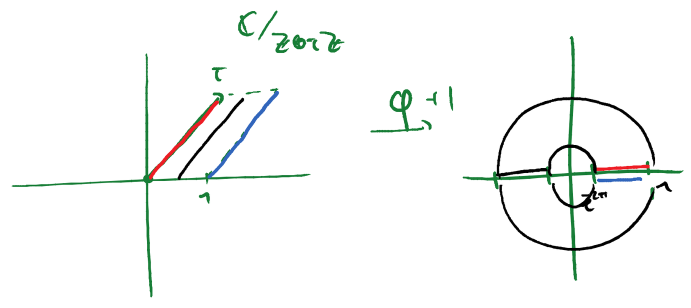
$\Rightarrow tr|_M\, Y_M(a,z)\, q^{L_0} = \sum_m \underbrace{tr|_M (a_m q^{L_0})}\, z^{-m-1}$

graded trace $tr|_V(a) = \sum_k tr|_{V_k}(a)\, q^k$

BUT: We want to work on the torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$!

$\Rightarrow$ coordinate transformation: $\varphi: \mathbb{C} \longrightarrow \mathbb{C}\setminus\{-1\}$ ← changes the global topology $\to$ not Möbius!

$z \longmapsto e^{2\pi i z} - 1$



(1) transforms as

(2) $F((a,z),q) := e^{2\pi i z \cdot \deg a}\, tr|_M\, Y(a, e^{2\pi i z})\, q^{L_0 - \frac{c}{24}}$ (torus 1-pt function)

We can pullback the VOA $(V, Y(\ ,z), \omega, |0\rangle)$ on the sphere along φ in order to get a new VOA on the cylinder $(V, Y[\ ,z], \tilde{\omega})$, where

(3) $Y[v,z] = e^{2\pi i z \deg v}\, Y(v, \varphi(z)) = \sum v[n]\, z^{-n-1}$

$\tilde{\omega} = \omega - \frac{c}{24}|0\rangle$

> Remember
>
> the partition function $Z(\tau)$ is the trace over the "propagator" on the cylinder
>
> $Z(\tau) = tr_H \exp(2\pi i(is P - tH))$ (↙ shift ↙ infinitesimal time)
>
> $= tr_H \exp(2\pi i(\tau L_0' - \bar{\tau}\bar{L}_0'))$, where $\tau = t + is$
>
> We have $L_0' = L[0] - \frac{c}{24}$
>
> L_0' defined by $Y(\omega,z) = -\sum L_n' e^{2\pi i n \varphi(z)}$

We want to take a closer look at $F((a,z),q)$:

$$F_M((a,z),q) = e^{2\pi i z \deg a} \sum_m tr|_M\, a_m\, e^{2\pi i z(-m-1)}\, q^{L_0 - \frac{c}{24}}$$

$$\overset{tr=0 \text{ for non degree preserving}}{=} tr|_M\, o(a)\, q^{L_0 - \frac{c}{24}}, \text{ where } o(a) = a_{\deg a - 1} \text{ is the zero mode (degree preserving)}$$

$\Rightarrow F((a,z),q)$ depends only on the zero mode of a and not on z!

In analogy to (2), we can define n-pt functions on the torus and a general recursion formula, coming from the locality of the VOA and the cyclicity of the trace, leads to:

$P_1(z,\tau) := \sum_{n\in\mathbb{Z}}' \frac{e^{2\pi i z n}}{1-q^n} - \frac{1}{2}$

(4) $$F_M((a,z),(b,w),q) = tr|_M\, o(a)o(b)\, q^{L_0 - \frac{c}{24}} - \sum_{m\geq 1} \frac{(-1)^m}{m!} P_1^{(m)}(z-w,q)\, F_M(o(a[m]b),q)$$

As they live on the torus, n-pt fctn's should be elliptic, i.e. periodic wrt the lattice $2\pi i\cdot(\mathbb{Z}+\tau\mathbb{Z})$

Let M_τ denote the set of all meromorphic elliptic functions wrt $2\pi i(\mathbb{Z}+\tau\mathbb{Z}) = \Lambda_\tau$

It turns out that $M_\tau = \mathbb{C}[\wp(z,\tau), \wp'(z,\tau)]$, where

(5) $$\wp(z,\tau) := \frac{1}{z^2} + \sum_{\omega\in\Lambda_\tau\setminus\{0\}}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$ (Weierstrass function)

For the canonical $SL(2,\mathbb{Z})$-action $\underbrace{\begin{pmatrix} a & b \\ c & d\end{pmatrix}}_{\sigma}.(z,\tau) = \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$ we obtain $\wp(\sigma.(z,\tau)) = (c\tau+d)^2\wp(z,\tau)$.

Moreover, we have $\wp(z,\tau) = \frac{1}{z^2} + \sum_{k\geq 2}(2k-1)E_{2k}(\tau)z^{2k-2}$

Here E_{2k} are the Eisenstein series. It is known that E_4 and E_6 span all modular forms of (i.e. $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$). E_2 is quasi-modular.

$P_1^{(k)}$ can be written as derivatives of the Weierstrass fctn $\Rightarrow$ elliptic!

Thus, 2-pt function elliptic if 1-pt fctn is elliptic ($\Leftrightarrow$ converges)

and $\mathrm{tr}\, o(a)o(b)q^{L_0-\frac{c}{24}}$ is elliptic ($\Leftrightarrow$ converges).

By associativity, we also have $F_M((a,z),(b,\omega),q) = \sum_n F_M(a[-n]b,\tau)(z-\omega)^{-n-1}$

In (4), we obtain

coefficients in $\mathbb{C}[E_2,E_4,E_6]$

(6) $$F_M(o(a[-1]b),q) = \mathrm{tr}_M\, o(a)o(b)q^{L_0-\frac{c}{24}} + \sum_{k\geq 1}E_{2k}(q)\, F_M(o(a[2k-1]b),q)$$

Lemma 1: Define $O_q(V)$ as the space generated by all states $v\in V$ of the form

$$v = a[-2]b + \sum_{k\geq 2}(2k-1)E_{2k}(\tau)\, a[2k-2]b$$

(in particular, $O_q(V) \subseteq \underbrace{V[E_4,E_6]}_{=V\otimes\mathbb{C}[E_4,E_6]}$)

$\Rightarrow F_M(o(v),q) = 0$.

pf: Set $a = L[-1]\tilde{a} = (2\pi i)^2(L_{-1}+L_0)\tilde{a} \Rightarrow o(a) = 0$.

Plug in a in (6) $\Rightarrow$

$$0 = \mathrm{tr}_M\, o(a)o(b)q^{L_0-\frac{c}{24}}$$

$k=1$-term vanishes, since $\mathrm{tr}\, o(\tilde{a}[0]b) = 0$ by cyclicity

$$= F_M(o(\underbrace{a[-1]b}_{\tilde{a}[-2]b}),q) + \sum_{k=2}(2k-1)E_{2k}(\tau)F_M(o(\tilde{a}[2k-2]b),q)$$

$\square$

Later, we proof:

Lemma 2: If V is C_2-cofinite, then for every $a\in V$ $\exists\, s\in\mathbb{N}$, $g_i\in\mathbb{C}[E_4(q),E_6(q)]$, s.t.

(7) $$L[-2]^s a + \sum_{i=0}^{s-1}g_i(q)\cdot L[-2]^i a \in O_q(V)$$

Lemma 3: Let $a \in V$ be a <u>highest weight vector</u> for the Virasoro algebra, then

$$(8) \qquad F\left(L[-i_1] \dots L[-i_k]a, q\right) = \sum_{i=0}^{n} g_i(q) \left(q \frac{d}{dq}\right)^i F(a,q)$$

for $g_i(q) \in \mathbb{C}[E_2(q), E_4(q), E_6(q)]$ this is proven by induction, using (6)

Combining lemmas 1, 2 and 3, we obtain a regular differential eq'n for $F_M(a,q)$.

$$(9) \qquad \left(\left(q\frac{d}{dq}\right)^s + \sum_{i=0}^{s-1} h_i(q) \left(q\frac{d}{dq}\right)^i\right) F_M(a,q) = 0$$

- $\in \mathbb{C}[E_2(q), E_4(q), E_6(q)]$
- converges on every closed subset of $|q|<1$
- depends on a but not on M

$$\Rightarrow \boxed{F_M\left(o(a), e^{2\pi i \tau}\right) \in \mathrm{Hol}(\mathbb{H})}$$

Example: (Yang-Lee) We look at the YL-minimal model at level $c = -\frac{22}{5}$.

It has two irreducible rep's, the vacuum $(h=0)$ and a rep at $h = -\frac{1}{5}$.

A null vector for the vacuum rep'n is given by

$$0 = N = \left(L[-4] - \tfrac{5}{3} L_{[-2]}^2\right)|0\rangle \in O_q(V)$$

Note that we want to derive the differential eq'n for <u>both</u> characters by just using <u>one</u> highest weight vector!

For the minimal models, we always have $L[-4]|0\rangle \in O_q(V)$ and hence we want to find the eq'n corresponding to $L_{[-2]}^2|0\rangle \in O_q(V)$.

Using lemma 3, we obtain

$$F\left(L[-2]^r|0\rangle, q\right) = P_r(D)\, F(|0\rangle, q)$$

where $D = q\frac{d}{dq}$ and, $P_r(D) \in \mathbb{C}[E_4, E_6][D]_r$

Computing $P_2(D) = (2\pi i)^4 \left(D^2 + \frac{c}{1440} E_4(q)\right) = (2\pi i)^4 \left(D^2 + \frac{11}{3600} E_4(q)\right)$

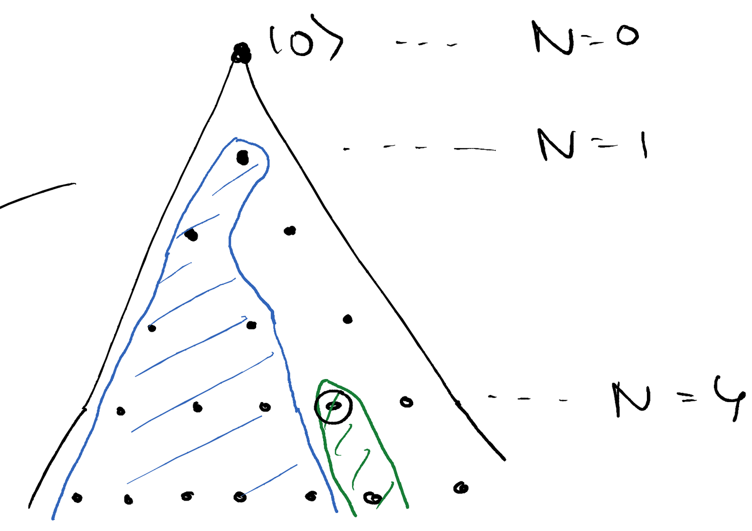
$$0 = F\left(L[-2]^2|0\rangle, q\right) = (2\pi i)^4 \left[\left(q\frac{d}{dq}\right)^2 - \frac{11}{3600} E_4(q)\right] Ch_0(q)$$

Two characters of YL $\leftrightarrow$ two solutions of (6), namely

$$Ch_0(q) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{\frac{(20n-3)^2}{40}} - q^{\frac{(20+7)^2}{40}}\right) = q^{-\frac{c}{24}} \underbrace{\frac{q^{\frac{1}{24}}}{\eta(q)}}_{=\sum p(n) q^n} \sum_n \left(q^{10n^2-3n} - q^{10n^2+7n+1}\right)$$

degree of Nullvector

$n=-1 \to (7,4)$



$$Ch_{-\frac{1}{5}}(q) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{\frac{(20n-1)^2}{40}} - q^{\frac{(20n+9)^2}{40}}\right), \text{ where}$$

$\eta(q) = q^{\frac{1}{24}} \prod (1-q^n)$ is the Dedekind eta function.

§2. C_2-cofiniteness

Remember that above, we claimed that if V is C_2-cofinite, then for every $a \in V$, we find $s, g_i(q), s.t.$

$$(7) \qquad L[-2]^s a + \sum_{i=0}^{s-1} g_i(q) \cdot L[-2]^i a \in O_q(V).$$

Remember that V is C_2-cofinite $\iff$ $V/C_2(V)$ fin. dim., where $C_2(V) := \langle a(-2)b \rangle$

By definition, $L[-i_1] \cdots L[-i_k] a$ form a basis for highest weight vectors a.

Lemma 4: If V is C_2-cofinite then $V[E_4(q), E_6(q)] / O_q(V)$ is a finitely generated $\mathbb{C}[E_4, E_6]$-module. $\square$

In particular, it is noetherian, and hence every submodule is finitely gen.
In particular, the submodules generated by $L[-2]^i a$, $i \in \mathbb{N}$.

$$\boxed{\Rightarrow C_2\text{-cofiniteness implies (7)}}$$

§3. Torus 1-pt conformal blocks:

Note that one actually has to define a system of n-pt CB's, leading to a conformal block!

Dfn: A map $S : V \otimes \mathbb{C}[E_4, E_6] \times \mathbb{H} \to \mathbb{C}$ is called a 1-pt. CB on the torus if

(i) $S(a, \tau)$ is holomorphic in $\tau \in \mathbb{H}$

(ii) $S(\sum a_i \otimes f_i, \tau) = \sum f_i(\tau) S(a_i, \tau)$

(iii) $S(a, \tau) = 0$ for $a \in O_q(V)$

(iv) $\forall a \in V_{[k]}$: $S(L[-2]a, \tau) = (2\pi i)^2 q \frac{d}{dq} S(a, \tau) + \sum_{k=1}^{\infty} \underbrace{G_{2k}(\tau)}_{= 2\zeta(k) E_{2k}(\tau)} S(L[2k-2]a, \tau)$

$\uparrow$ grading wrt $L[0]$

From the dfn it is already clear that 1-pt CB's on the torus behave similar as 1-pt functions on the torus $F(o(a), q)$.

In fact, in the previous sections, we have actually proven that

$S_M(a, \tau) := F(o(a), e^{2\pi i \tau})$ defines a 1-pt conformal block if V is C_2-cofinite.

One of the main motivations for the above definition is the following theorem:

Thm: [Zhu] Does it for n-pt fctn's

Let S be a 1-pt CB on the torus and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$

$\Rightarrow (\sigma . S)(v, \tau) := (c\tau + d)^{-\deg a} S\left(v, \frac{a\tau + b}{c\tau + d}\right)$ is a 1-pt fctn on the torus.

$\Rightarrow (\sigma . S)(v,\tau) := (c\tau + d)^{-\mathrm{wt}\,v} \, S\left(v, \frac{a\tau + b}{c\tau + d}\right)$ is a 1-pt fctn on the torus.

$\Rightarrow SL(\mathbb{Z},\mathbb{Z})$ acts on space of 1-pt CB's

<u>pf</u>: All clear, except for (iii) & (iv)

<u>Thm</u>: Let V be C_2-cofinite

- the space of 1-pt fctn's is finite dim.
- if V is also rational, the torus 1-pt fctn's have a basis S_{M_i} for $\{M_i\}$ complete list of irrep's