Conformal field theories

Conformal transformations

Def: Let (M,g) and (M',g') Riemannian manifolds of dim. n and let $U \subset M$, $V \subset M'$ open subsets. A smooth mapping $\varphi : U \to V$ of max. rank is called a conformal transformation, if there is a smooth fct. $\Omega : U \to \mathbb{R}_+$, called conformal factor, s.th.

$$\varphi^* g' = \Omega^2 g \qquad \text{or} \qquad \partial_\mu \varphi^\rho \partial_\nu \varphi^\sigma g'_{\rho\sigma} = \Omega^2 g_{\mu\nu} \quad \text{(Einstein notation)}$$

where $\varphi^* g'(X,Y) := g'(D\varphi(X), D\varphi(Y))$.

Sometimes: require that φ bijective and/or orientation preserving

preserve angles: $\cos \alpha = \dfrac{\varphi^* g'(X,Y)}{\sqrt{\varphi^* g'(X,X)}\sqrt{\varphi^* g'(Y,Y)}} = \dfrac{\Omega^2 g(X,Y)}{\Omega^2 \sqrt{g(X,X)}\sqrt{g(Y,Y)}}$

Examples:
- local isometries with $\Omega = 1$ ($\varphi^* g' = g$), i.e. translations & rotations in $\mathbb{R}^n$
- Euclidean plane $\mathbb{R}^2 \simeq \mathbb{C}$ via $z = x + iy$, $(x,y) \in \mathbb{R}^2$
  $\varphi : U \to \mathbb{C}$ conformal $\Leftrightarrow$ φ satisfies Cauchy-Riemann-eq. (with $|D\varphi| \neq 0$)
  $\Leftrightarrow$ φ holomorphic or antiholomorphic with $\Omega = |D\varphi| \neq 0$
- dilation in $\mathbb{R}^n$: $\varphi(x) = \lambda x$, $\Omega = \lambda > 0$

Conformal Killing vector fields

Idea: classify conformal maps by (conformal Killing) vector fields (CKV)

From now on: just $\mathbb{R}^n$

Let $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ smooth vf with flow φ_t: $\frac{d}{dt}\varphi_t(a) = X(\varphi_t(a))$

Def: X is called conformal Killing vf, if φ_t is a conformal transformation for all t in a neighbourhood of 0.

Thm: Let $X = (X^1, \dots, X^n) = X^\nu \partial_\nu$ be a conf. Killing vf on $U \subset \mathbb{R}^n$

Then $\exists$ smooth $\kappa : U \to \mathbb{R}$ s.th.

$$X_{\mu,\nu} + X_{\nu,\mu} = \kappa g_{\mu\nu} \qquad (*)$$

where $f_{,\nu} := \partial_\nu f$, $X_\mu := g_{\mu\nu} X^\nu$

This corresponds to a infinitesimal coord. transformation "$x^\mu \to x^\mu + X^\mu$"

Def: κ <u>conf. Killing factor</u> $\Leftrightarrow$ $\exists$ CKV s.th. $(*)$ holds.

Thm: κ conf. Killing factor $\Leftrightarrow$ $(n-2)\kappa_{,\mu\nu} + g_{\mu\nu} \Delta \kappa = 0 \qquad (**)$

$\rightsquigarrow$ $n=2$ special: $\Delta \kappa = 0$

$n > 2$: $(**)$ holds componentwise $\Rightarrow$ additional constraints

Case 1: $\mathbb{R}^n$, $n > 2$

Thm: Every conf. transformation $\varphi : U \to \mathbb{R}^n$, U connected open subset of $\mathbb{R}^n$, is a composition of

- translations: $q \mapsto q + c$, $c \in \mathbb{R}^n$
- orthogonal transformation: $q \mapsto \Lambda q$, $\Lambda \in O(n)$
- dilatation: $q \mapsto \lambda q$, $\lambda \in \mathbb{R}$
- special conformal transformations $q \mapsto \dfrac{q - \langle q,q \rangle b}{1 - 2\langle q,b \rangle + \langle q,q \rangle \langle b,b \rangle}$, $b \in \mathbb{R}^n$

Note: for a given $b \in \mathbb{R}^n$, the special conformal transformation is not everywhere well-defined! Solution: next chapter: add point at ∞

Case 2: $\mathbb{R}^2$

Thm:
- The orientation-preserving conformal transformations $\varphi : U \to \mathbb{R}^2$, $U \subset \mathbb{R}^2$ open & connected, are exactly the holomorphic fcts. with $|D\varphi| \neq 0$. Every CKV $X : U \to \mathbb{C}$ is holomorphic.
- Likewise: orientation-reversing $\leftrightarrow$ antiholomorphic
- holom. & antiholom. maps exhaust conf. trans. on U

Conformal compactification & conformal group

Def: The conformal group $Conf(\mathbb{R}^n)$ is the connected component containing the identity in the group of conformal diffeo. of the "conformal compactification" (see below) of $\mathbb{R}^n$.

Case 1: $\mathbb{R}^n$, $n>2$

Thm: Every conformal transformation on an open connected subset $U \subset \mathbb{R}^n$, $n>2$, has a unique conf. continuation to S^n ("conf. compactification"). The group of all conf. transf. $S^n \to S^n$ is isomorphic to $O(n+1,1)/\{\pm 1\}$

$Conf(\mathbb{R}^n) = SO^+(n+1,1)$

Idea: Embed $\iota: \mathbb{R}^n \hookrightarrow \mathbb{R}P^{n+1}$, $S^n \simeq \overline{\iota(\mathbb{R}^n)}$, with projection $\pi: \mathbb{R}^{n+2,1} \setminus \{0\} \to \mathbb{R}P^{n+1}$

CKV extend to global vf., integrate to conf. maps

Case 2: $\mathbb{R}^2$

many non-injective holomorphic maps: $z \mapsto z^k$, $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, $z \in \mathbb{C} \setminus \{0\}$

injective, without holom. continuation: $z \mapsto \sqrt{z}$, $z \in \{w \in \mathbb{C} \,/\, \mathrm{Re}\, w > 0\}$

or: $z \mapsto \log z$, $z \in \mathbb{C} \setminus \{-x \,/\, x \in \mathbb{R}_+\}$

Def: global conf. transformation on $\mathbb{R}^2 \simeq \mathbb{C}$ $:\Leftrightarrow$ injective holom. transf. on $\mathbb{C} \setminus \{p\}$ with at most one exceptional point $p \in \mathbb{C}$

Thm: Every global conf. trafo φ on $U \subset \mathbb{C}$, U open & connected, has a unique conf. continuation $\hat{\varphi}: S^2 \to S^2$. The group of conf. diffeo. is isomorphic to $O(3,1)/\{\pm 1\}$ and $Conf(\mathbb{R}^2) \cong SO^+(3,1)$

Def: A Möbius transf. is a holom. fct. φ, for which there is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ s.th. $\varphi(z) = \frac{az+b}{cz+d}$, $cz+d \neq 0$

The set of Möbius transf. $\mathcal{M}b$ is precisely the set of all orientation preserving global conf. transf. on $\mathbb{C}$.

$$\mathcal{M}b \cong PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\{\pm 1\} = Aut(S^2) \cong SO(3,1) \cong Conf(\mathbb{R}^2)$$

Witt algebra

conf. transf. on $U \subset \mathbb{C}$, open & connected, are holom. or antiholom. fcts. φ with $|D\varphi| \neq 0$. Infinitesimal holom. trafos:

$$z \mapsto z + \varepsilon(z) = z + \sum_{n \in \mathbb{Z}} a_n z^n$$

corresponding vf representing this infinitesimal transfs

$$\sum_{n \in \mathbb{Z}} a_n z^{n+1} \frac{d}{dz}$$

Lie algebra of vf. has (topological) basis $L_n = z^{1-n} \frac{d}{dz}$ with

$$[L_n, L_m] = (n-m) L_{n+m}$$

Witt algebra: $W = \mathrm{span}_{\mathbb{C}} \{ L_n \mid n \in \mathbb{Z} \}$ as conf. symmetry algebra

Notes:
- It is a dense subalgebra of holom. vf on $\mathbb{C} \setminus \{0\}$
- L_n, $n > 1$, strictly singular at 0 (correspond to proper deformations of conf. structure on $\mathbb{C}$)
- later: "primary fields" or "heighest weight states" are annihilated by all L_n, $n > 1 \Rightarrow$ infinite number of constraints
- $\nexists$ complex Lie Group H with $\mathrm{Lie}\, H \cong \mathrm{Vect}^{\mathbb{C}}(S^1) = \hat{W}$, where $\hat{W}$ denotes the closure of W
- second copy of W, denoted $\overline{W}$, for antiholom. vf.
- for $z \mapsto 1/w$, $L_n \mapsto -w^{n+1} \frac{d}{dw}$

Correlators of 2d CFTs

correlators of primary fields: ~~[illegible]~~

$$\langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = \left(\frac{\partial f}{\partial z}\right)^{h_1}_{z=z_1} \cdots \left(\frac{\partial f}{\partial z}\right)^{h_n}_{z=z_n} \langle \phi_1(f(z_1)) \dots \phi_n(f(z_n)) \rangle \quad (***)$$

for all holom. maps f (i.e. conf. transf.). For simplicity, we omit any antiholom. dependency. $h_i \in \mathbb{R}$ "conformal weights"

$(***)$ imposes serve constrains on correlators!

2-pt. fct.: $\langle \phi_i(z_1) \phi_j(z_2) \rangle = \dfrac{C_{ij}}{z_{12}^{2h}}$, $z_{12} = z_1 - z_2$, $h = h_i = h_j$,

C_{ij} is a normalization constant, can be chosen δ_{ij}

3-pt. fct.: $\langle \phi_i(z_1)\,\phi_j(z_2)\,\phi_k(z_3)\rangle = \overset{\mathbb{C}}{\overset{\cup}{C_{ijk}}}\; z_{12}^{h_k-h_i-h_j}\, z_{23}^{h_i-h_j-h_k}\, z_{13}^{h_j-h_i-h_k}$

4-pt. fct.: $\langle \phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)\,\phi_4(z_4)\rangle = f(x)\prod_{i<j} z_{ij}^{h/3-h_i-h_j}$, $h=\sum h_i$

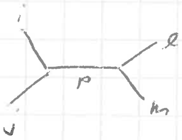
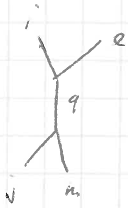
$x = \dfrac{z_{12}\,z_{34}}{z_{13}\,z_{24}}$ "conformal cross ratio", f holom. map

$\to$ 4-pt. fcts. & higher are not completely fixed by conf. invariance!

<u>However</u>: Bootstrap: The order of OPE should not matter, i.e.

$$\langle \phi_i(z_1)\,\phi_j(z_2)\,\phi_k(z_3)\,\phi_l(z_4)\rangle$$

OPE $z_1\to z_2$, $z_3\to z_4$ | OPE $z_1\to z_3$, $z_2\to z_4$

$\sum_p C_{ijp}\,C_{lmp}$  $\overset{!}{=}$ $\sum_q C_{ilq}\,C_{jmq}$  $(****)$

(here we used the normalization $C_{ij}=\delta_{ij}$ of 2pt-fct.)

$\leadsto$ infinite number of constrain for C_{ijk}

$(****)$ is called <u>crossing symmetry</u>, and defines <u>associativity-like</u> constraints on the OPEs.

More on this in later talks!