Symmetries of a quantum theory

Now, want a quantum theory with conformal space-time symmetry. That means, we need a state space, which is a Hilbert space, $\mathbb{H}$. However, in quantum theory one only has normalised states, and hence we actually want the projective space (space of all 1-dimensional subspaces of $\mathbb{H}$)

$$\mathbb{P} := \mathbb{H}\setminus\{0\}/\sim \qquad (x \sim y \text{ iff } \exists \lambda \in \mathbb{C} \; x = \lambda y)$$

Why normalised? Because the only measurable things are transition probabilities (which are invariant under $f \to \lambda f$, $g \to \mu g$) (or simply "overlaps")

to avoid using any notion of time ...

$$\delta(\varphi, \psi) := \frac{|\langle f, g\rangle|^2}{\|f\|^2 \|g\|^2} \qquad , f, g \in \mathbb{H}$$

where

$$\varphi := \gamma(f)$$
$$\psi := \gamma(g)$$

and

$$\gamma : \mathbb{H}\setminus\{0\} \longrightarrow \mathbb{P} \qquad \text{the projection}$$

So we want the projective space to be invariant under the conformal symmetry transformations.

Def: A bijective map $T : \mathbb{P} \to \mathbb{P}$ with the property

s.t. $\left(\delta(T\varphi, T\psi) = \delta(\varphi, \psi)\right) \qquad \varphi, \psi \in \mathbb{P}$

is called a projective automorphism. The group of all such maps is called $\mathrm{Aut}(\mathbb{P})$. the same!

Def: A symmetry of a quantum system is a bijective trafo on $\mathbb{P}$ leaving δ invariant.

Hence, $\mathrm{Aut}(\mathbb{P})$ is the full symmetry group of the quantum mechanical phase space. What we want is a group homomorphism

$$R : \underset{\mathrm{Conf}_d}{\underset{\cup}{G}} \longrightarrow \mathrm{Aut}(\mathbb{P})$$

But: We usually do all calculations in $\mathbb{H}$, hence have to understand how to "pull back" the symmetry along γ.

Lifting symmetries to $\mathbb{H}$

What is the analog of symmetry-preserving operations on $\mathbb{H}$?

Def: A unitary operator U on $\mathbb{H}$ is a $\mathbb{C}$-linear bijective map $U: \mathbb{H} \longrightarrow \mathbb{H}$ leaving the inner product invariant:

$$f, g \in \mathbb{H} \implies \langle Uf, Ug \rangle = \langle f, g \rangle$$

The group of all such operators is called the unitary group of $\mathbb{H}$; denoted by $U(\mathbb{H})$.

Can also define anti-unitary operators. These are $\mathbb{R}$-linear and satisfy $\langle Vf, Vg \rangle = \overline{\langle f, g \rangle}$, $V(if) = -iV(f)$.

Every unitary operator U induces a projective automorphism $\hat{\gamma}(U) \in \mathrm{Aut}(\mathbb{P})$ via

$$\hat{\gamma}(U)(\underset{\overset{\shortparallel}{\gamma(f)}}{\varphi}) := \gamma(U(f))$$

$$\begin{tikzcd} \mathbb{H} \arrow[r, "U"] \arrow[d, "\gamma"'] & \mathbb{H} \arrow[d, "\gamma"] \\ \mathbb{P} \arrow[r, "\hat{\gamma}(U)"'] & \mathbb{P} \end{tikzcd}$$

- well-def'd

it is actually a symmetry

- $\delta(\hat{\gamma}(U)\varphi, \hat{\gamma}(U)\psi) = \delta(\gamma(U(f)), \gamma(U(g)))$

$$= \frac{|\langle Uf, Ug \rangle|^2}{\|Uf\|^2 \, \|Ug\|^2} = \frac{|\langle f, g \rangle|^2}{\|f\|^2 \, \|g\|^2} = \delta(\varphi, \psi)$$

$$\implies \hat{\gamma}(U) \in \mathrm{Aut}(\mathbb{P})$$

- $\hat{\gamma}$ is a group homomorphism

The same can be done for anti-unitary operators.

The following theorem tells us that the converse holds as well:

Wigner's Theorem: Every projective transformation $T \in \mathrm{Aut}(\mathbb{P})$ is induced by a unitary or an antiunitary operator U via $T = \hat{\gamma}(U)$.

Def: $U(\mathbb{P}) := \hat{\gamma}(U(\mathbb{H})) \subset \mathrm{Aut}(\mathbb{P})$

(Then $U(\mathbb{P}) \cup \underset{\text{anti-unitary}}{\hat{U}(\mathbb{P})} \underset{?!}{=} \mathrm{Aut}(\mathbb{P})$)

So Wigner tells us that instead of working with symmetries on $\mathbb{P}$, we can consider unitary operators on $\mathbb{H}$.

Image in $U(\mathbb{P})$ because normally want G topological & R continuous. Then, since $U(\mathbb{P})$ and $\tilde{U}(\mathbb{P})$ are disconnected components and $\tilde{U}(\mathbb{P})$ not closed $\Rightarrow$ im $R \subset U(\mathbb{P})$

But for any group homomorphism

$$R : G \longrightarrow U(\mathbb{P}) \subset \mathrm{Aut}(\mathbb{P}),$$

($\hat{=}$ "projective repr$^{\underline{n}}$ τ of G)

do we find a group homomorphism

$$\rho : G \longrightarrow U(\mathbb{H})$$

($\hat{=}$ "unitary repr$^{\underline{n}}$ of G)

s.t.

$$\begin{tikzcd} & G \arrow[dl, "\rho"'] \arrow[d, "R"] \\ U(\mathbb{H}) \arrow[r, "\hat{\gamma}"'] & U(\mathbb{P}) \end{tikzcd}$$

?

<u>General answer</u>: No! But:

primitive version of Bargmann's theorem →

<u>Theorem</u>: Let G be a group and $R : G \longrightarrow U(\mathbb{P})$ a homomorphism. Then there exists a unique central extension E of G by $U(1)$ and a homomorphism $\tilde{\rho} : E \longrightarrow U(\mathbb{H})$ s.t.

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d, "\tilde{\rho}"] & G \arrow[r] \arrow[d, "R"] & 1 \\ 1 \arrow[r] & U(1) \arrow[r] & U(\mathbb{H}) \arrow[r, "\hat{\gamma}"'] & U(\mathbb{P}) \arrow[r] & 1 \end{tikzcd}$$

commutes.

"almost a reprn of G", but only "up to $U(1)$" (i.e. up to a constant)

Note: There is always a linear map $\rho : G \longrightarrow U(\mathbb{H})$, but we can see that it is only a homomorphism if $1 \to U(1) \to E \to G \to 1$ splits, i.e. iff $E = G \overset{\times}{\oplus} U(1)$

<u>Takeaway</u>: A "physical" symmetry G is replaced by a central extension on the Hilbert space.

<u>Note</u>: Can replace "group" by "Lie algebra" in the above discussion.

→ In our case: Need to consider Hilbert spaces with actions of central extensions of the Witt algebra

Def: The Virasoro algebra is the central extension

$$1 \longrightarrow \underset{\text{Lie}(U(1))}{\mathbb{C}} \longrightarrow \mathrm{Vir} \longrightarrow \mathrm{Witt}_{\mathbb{C}} \longrightarrow 1$$

($\mathrm{Vir} = \mathrm{Witt} \oplus Z\mathbb{C}$ as vector space over $\mathbb{C}$) with

note: take complexified Lie Algebra because easier to work with. Later reverse by setting $L_n^+ = L_{-n}$

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0} \frac{n}{12}(n^2-1)Z$$

$$[L_n, Z] = 0$$

$$n, m \in \mathbb{Z}.$$

Thm: Vir is the only central extension of $\mathrm{Witt}_{\mathbb{C}}$ by $\mathbb{C}$.
(Proof: $H^2(g,a) \overset{1:1}{\longleftrightarrow} \{\text{central extensions } a \to e \to g\}$. Can compute $H^2(\mathrm{Witt}, \mathbb{C}) \cong \mathbb{C}$)

(Aside) Note: As for the Witt algebra, $\nexists$ complex Lie group VIR, s.t. $\mathrm{Lie}(\mathrm{VIR}) = \mathrm{Vir}$

Note: $\langle L_{-1}, L_0, L_1 \rangle$ (Möbius) doesn't feel the extension! actually: $\mathrm{Vir} \times \mathrm{Vir}$

Now we want to build Hilbert spaces with Virasoro-symmetry.
Hence; study:

Representation Theory of Vir

Note: Slightly non-standard def. Normally require definite H

Def: A repr$^{\underline{n}}$ ρ : Vir $\longrightarrow$ End V (vsp. over $\mathbb{C}$) is called unitary if there is a positive semi-definite hermitian form

$$H : V \times V \longrightarrow \mathbb{C}$$

$\langle v,v \rangle \geq 0 \ \forall v$ (bilinear & $\langle v,w \rangle = \overline{\langle v,w \rangle}$)

s.t. $H(\rho(L_n)v, w) = H(v, \rho(L_{-n})w)$ & $H(\rho(Z)v, w) = H(v, \rho(Z)w)$

Same as saying $L_n^+ = L_{-n}$

Note: In fact, we want V to be part of a Hilbert space, so we will need $\langle v,v \rangle = 0 \iff v = 0$ (i.e. H a true inner product). But it is much easier to study the possible representations in more generality, and later we will throw away "garbage" we don't want (i.e. null states v with $\langle v,v \rangle = 0$)

We have another physically motivated type of representations:

<u>Def</u>: A repr$^{\text{n}}$ $\rho: \mathrm{Vir} \to \mathrm{End}\, V$ is called a <u>highest weight repr$^{\text{n}}$</u> if there are complex numbers $h, c \in \mathbb{C}$ and a cyclic vector $v_0 \in V$ s.t.
(highest w.v.)

$v_0 \in V$ s.t. $\rho(\mathrm{Vir}) v_0 = V$

$$\rho(Z) v_0 = c\, v_0$$
$$\rho(L_0) v_0 = h\, v_0$$
$$\rho(L_n) v_0 = 0 \qquad \forall n \geq 1$$

Physicist notation:
$c|h\rangle = c|h\rangle$
$L_0|h\rangle = h|h\rangle$
$L_n|h\rangle = 0$

v_0 is then called highest-weight vector and (V, ρ) a <u>Virasoro module</u> for h, c.

<u>Motivation for this def$^{\text{n}}$</u>: $L_0 \sim$ dilatation generator $\sim$ energy
explain !?
Want have a state space where energy spectrum is diagonalisable and bounded from below.
If we assume this, and v_0 is the lowest eigenvalue h, then must have $\rho(L_n) v_0 = 0 \quad \forall n \geq 1$:
exception: LogCFT!

$$L_0 (L_n|h\rangle) = L_n L_0 |h\rangle - n L_n|h\rangle = (h-n) L_n |h\rangle$$
$\longrightarrow$ either $L_n|h\rangle = 0$ or the "energy" of $L_n|h\rangle$ is $< h$ $\lightning$

<u>Remark</u>: Virasoro modules are naturally graded:
$$V = \bigoplus_{N \in \mathbb{N}} V_N \quad \longleftarrow \quad \underline{\text{NOTE}}: \text{More generally in any dimension!}$$
where
$$V_0 = \mathbb{C} v_0, \quad V_N = \mathrm{span}\left\{\rho(L_{-n_1}) \cdots \rho(L_{-n_k}),\ \sum_{j=1}^{k} n_j = N\right\}$$
Note that $L_0 V_N = (N+h) V_N$.

<u>Def</u>: A Verma module $M(c,h)$ for $c, h \in \mathbb{C}$ is a Virasoro module where
$$\{\rho(L_{-n_1}) \cdots \rho(L_{-n_k}) v_0 : n_1 \geq \ldots \geq n_k > 0,\ k \in \mathbb{N}\} \cup \{v_0\}$$
is a basis (as opposed to just spanning the module)

<u>Lemma</u>: For every $c, h \in \mathbb{C}$, a Verma module $M(c,h)$ exists.

This makes H symmetric on basis vectors

Take from now on $c, h \in \mathbb{R}$

We can define a hermitean form on the basis of our Verma modules (and hence Virasoro modules):

$$H\left(L_{-n_1} \cdots L_{-n_k} v_0 , L_{-m_1} \cdots L_{-m_\ell} v_0\right)$$

$$:= \langle L_{n_k} \cdots L_{n_1} L_{-m_1} \cdots L_{-m_\ell} v_0 \rangle$$

where $\langle w \rangle$ for some $w \in M(c,h)$ is defined as the (unique) component of w in V_0. (the level 0 subspace)

Physics lingo: $= \langle h | L_{n_k} \cdots L_{n_1} L_{-m_1} \cdots L_{-m_\ell} | h \rangle$

This is reversing the complexification of Witt / Vir

where $L_{-n} =: L_n^+$, $\langle h | h \rangle = 1$

Can extend to whole $M(c,h)$ via $\left(w = \sum_j \lambda_j w_j , w' = \sum_k \mu_k w_k\right)$

$$H(w,w') := \sum \sum \bar{\lambda}_j \mu_k H(w_j, w_k')$$

<u>Thm</u>: Let $c, h \in \mathbb{C}$ and $M = M(c,h)$

1) H is the unique hermitean form on M satisfying $H(v_0, v_0) = 1$ and $H(L_n v, w) \overset{L_n = L_{-n}^+}{=} H(v, L_{-n} w)$ and $H(Zv, w) = H(v, Zw)$

2) $H(V_N, V_M) = 0$ for $N \neq M$ (so $V_N \perp V_M$)

3) $\ker H$ is the maximal proper submodule of M.

$\ker H = \{ v \in M \mid H(w,v) = 0 \ \forall w \in M \}$

Proof: On basis vectors v, w

1) $H(L_n v, w) =$ commutation relations

2) need $\sum_i^k n_i = \sum_j^\ell m_j$, otherwise commutation relations always end up annihilating everything. (Can always write $L_{n_k} \cdots L_{n_1} L_{-m_1} \cdots L_{-m_\ell} = \sum P_\ell v_0$

3) • $v \in \ker H \Rightarrow L_n v \in \ker H$: $(H(w, L_n v) = H(L_{-n} w, v) = 0)$ [$v \in \ker H$]

• $\ker H \neq M$: $(H(v_0, v_0) \neq 0 \Rightarrow v_0 \notin \ker H)$ $\ldots L_s$, $s > 0$

• $U \subsetneq M$ (submodule) $\Rightarrow U \subset \ker H$: need to think a bit ...

Remark: $M(c,h)/\ker H$ is a Virasoro module with nondegenerate hermitean form H. However, in general not definite (i.e. $\exists$ negative norm states)

Let's just calculate a bit:

$$H(L_n v_0, L_n v_0) = \langle h | L_n L_{-n} | h \rangle$$
$$\overset{\text{vir}}{=} \langle h | [L_n, L_{-n}] . | h \rangle$$

$[L_n, L_m] = (n-m) L_{n+m} + \delta_{n+m,0} \frac{c}{12} n(n^2-1)$

$$= \langle h | 2n L_0 + \frac{c}{12} n(n^2-1) | h \rangle$$
$$= 2nh + \frac{c}{12} n(n^2-1)$$

$\longrightarrow$ so for certain h there are n s.t. this becomes 0.

Also can see:

if H positive semidefinite $\Longrightarrow$ $c \geq 0$, $h \geq 0$

- $n=1$: $\langle \dots \rangle = 2h \Rightarrow h \geq 0$
- $\forall n$: $2nh + \frac{c}{12} n(n^2-1) \geq 0 \Rightarrow c \geq 0$

Using sth. called the Kac determinant, we can classify completely for which h, c the Verma-module $M(c,h)$ is unitary:

Thm: Let $c, h \in \mathbb{R}$.

1) $M(c,h)$ is unitary (positive definite) (semi) for $c > 1, h > 0$ ($\geq$, $\geq$)

2) $M(c,h)$ is unitary for $0 \leq c < 1$, $h > 0$ if and only if there exists some $m \in \mathbb{N}_{>0}$ and integers p, q $1 \leq p \leq q < m$ with

$$h = h_{p,q}(m) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$$

$$c = c(m) = 1 - \frac{6}{m(m+1)}$$

(to each value of $m = 3, 4, \dots$ there are $m(m-1)/2$ allowed values of h)

These give rise to the minimal models
$\rightsquigarrow$ strongly constrained: Hilbert space consists of finite numbers of Virasoro modules, Virasoro modules are simplified by the existence of null-states (because not positive definite)

Note: Factorising out all null states from a positive semi-definite $M(c,h)$ gives us a positive definite

$$\underset{\uparrow \atop \text{positive definite}}{W(c,h)} := \underset{\uparrow \atop \text{semi-definite}}{M(c,h)} \Big/ \ker H$$

(Also, for each $c, h \in \mathbb{R}$ there is at most one unitary h.w rep., which must be $W(c,h)$)

Thm: For each weight (c,h):
1) $M(c,h)$ is indecomposable
2) If $M(c,h)$ reducible $\Rightarrow \exists$ max. invariant subspace $I(c,h)$ s.t. $M(c,h)/I(c,h)$ is an irred h.w.m.
3) any positive definite unitary h.w.r. (i.e. $W(c,h)$) is irreducible

$\rightarrow$ Takeaway: We understand how to get irreducible representations that satisfy our physical criteria (bounded spectrum & unitarity)

Can now build state space for given c:

$$H = \bigoplus_{h,\bar{h}} V(c,h) \otimes V(c,\bar{h}) \qquad (\text{rep}^{\underline{n}} \text{ of } Vir \times Vir)$$

irreducible Virasoro modules as built above

- correlators
- state ↔ field
- OPE
- bootstrap prog.

Today, in those 30 mins, Yannik & me will present some key concepts of CFT, rather heuristically. The next speakers will then attempt to formalise this / work it out in more detail.

Last time we discussed how to find good building blocks of a Hilbert space with conformal symmetry (i.e. reps of Vir, unitary, highest weight)
in 2d, but some things are generalisable
Can then build a space

$$V = \bigoplus_{h,\bar{h}} V(c,h) \otimes V(c,\bar{h}) \qquad \leftarrow \text{rep of } Vir \times Vir$$

Have seen how the choice of c is limited and how it constrains the possible values of h & $\bar{h}$ appearing in the sum. But $\exists$ more consistency criteria on the set of $h, \bar{h}$ which we will not at the moment go into.
As our Hilbert space, we take

$$H = \overline{V}$$

Note: In subsequent talks (on VOAs & conformal blocks) will only consider the holomorphic part of the theory, which means we look at

$$V = \bigoplus_{h} V(c,h)$$

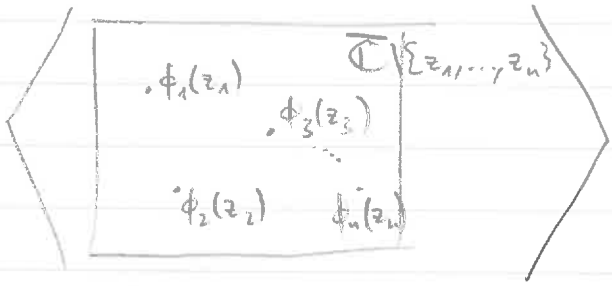
Idea is to "glue" holomorphic & antiholomorphic theory in a consistent way in the end.

Correlation functions (pictures in 2d)

We don't have time to go into details about what a field is. For the moment, it is an element in a vectorspace (which is a module of our desired symmetries) that we can associate to punctures on our space ($\overline{\mathbb{C}}$ for the moment), $H = \overline{V}$.

aside [We may see in the next talk briefly how a field can be understood as an operator as well.]

An n-point correlation function is a map

$\langle \phi_1(z_1) \dots \phi_n(z_n) \rangle =$ 

$$: \underbrace{H^{\otimes n}}_{\text{multilinear in these}} \times \underbrace{\mathbb{C}^n \setminus diag}_{\text{smooth in these}} \longrightarrow \mathbb{C}$$

which behaves covariantly under symmetry (here $z \to w$ conformal)
(i.e. $\langle \phi_1(z_1) \dots \rangle (dz_1)^{\Delta_1} \cdots (dz_n)^{\Delta_n} = \langle \phi_1(w_1) \dots \phi_n(w_n) \rangle (dw_1)^{\Delta_1} \cdots (dw_n)^{\Delta_n}$)
+ additional requirements (e.g. gluing / cutting)

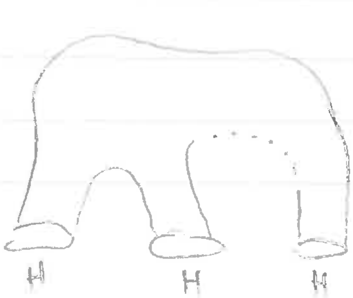
aside [These maps have an actual interpretation as correlations (expectation values of "products of fields in different points") in the sense learned in probability theory. This is reflected in the path integral approach

$$\langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = \frac{\int D\phi \; \phi_1(z_1) \cdots \phi_n(z_n) \, e^{iS[\phi]}}{\int D\phi \; e^{iS[\phi]}}$$

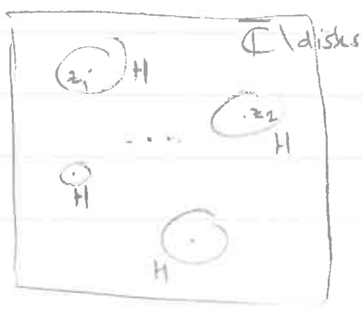
as well as the operator approach:

$$\langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = \langle \underset{\overset{\cap}{H}}{\phi_1(z_1)} | \underbrace{\dots\dots}_{\in \operatorname{End} H} | \underset{\overset{\cap}{H}}{\phi_n(z_n)} \rangle$$
]

People more familiar with TFTs might also think of a correlator as



$\rightsquigarrow$

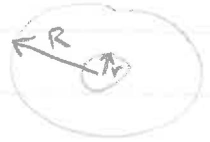


$\rightsquigarrow$ lin. map $H^{\otimes n} \times \mathbb{C}^n \longrightarrow \mathbb{C}$

Here, H is understood as the <u>state space</u> (states "live" on $d-1$ dim. submanifolds, fields "live" on points).

<u>Claim</u>: In CFT States $\cong$ Fields (<u>State-field correspondence</u>)

Argument goes like this: It is natural to map the annulus to the exponential of the dilatation operator

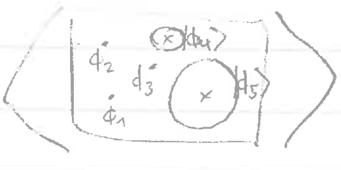


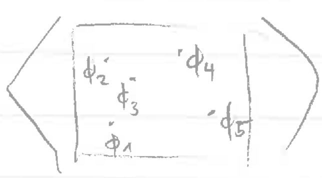
$$\longmapsto \quad H \longrightarrow H, \qquad \phi \longmapsto \left(\frac{r}{R}\right)^{L_0+\bar{L}_0}\phi$$

But for $\phi \in V \subset \bar{V} = H$, ϕ is an eigenvector of this map! ($(L_0+\bar{L}_0)\phi = \Delta\phi$) (this is specific to unitary CFT)

$$r^{-\Delta}\phi \longmapsto \left(\frac{r}{R}\right)^{L_0+\bar{L}_0} r^{-\Delta}\phi = R^{-\Delta}\phi \quad \in V$$

is a trivial isomorphism (on the level of V). So I might as well define a field to be a state that lives on an arbitrarily small circle. This motivates us to introduce also correlators with boundary via

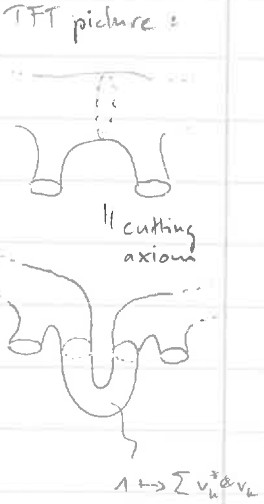
e.g. 

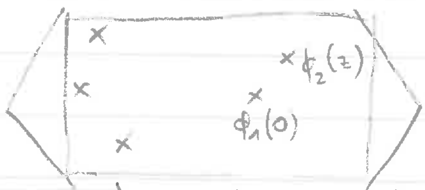
:= 

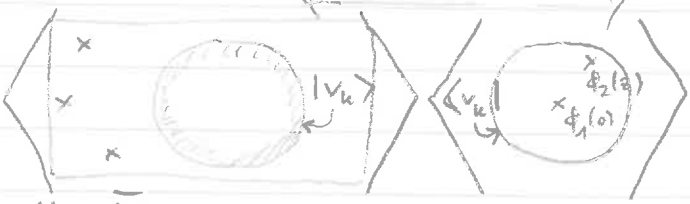
<u>Operator-Product expansion</u>

By employing some cutting-axiom (want this for all QFTs) we can decompose a correlator into:

TFT picture:



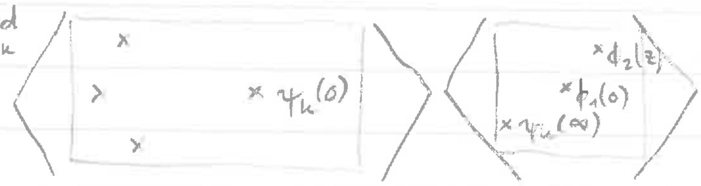
$\langle \phi_1(0)\phi_2(z) * \rangle =$ 

$= \sum_k$ 

$|v_k\rangle \in V$ orthonormal basis of $L_0+\bar{L}_0$-eigenstates

converges in $H = \bar{V}$

state $|v_k\rangle \overset{1:1}{\longleftrightarrow}$ field ψ_k

$= \sum_k$ 

$c_{12k}(z)$

$= \sum_k c_{12k}(z) \langle \psi_k(0) * \rangle$ $\longrightarrow$ converges up to nearest field insertion