TQFTs & Topological Quantum Computation, Part 2

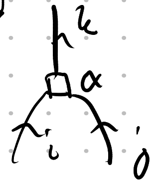
1) Turaev–Viro TQFT & String Net Models

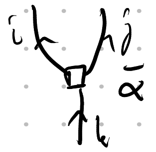
- [Wang; TQC]
- [Turaev, Vir: Monoidal Cat...]
- [Kir; 1106.6033]

We start by summarizing the TV construction.

Input: Spherical fusion category $\mathcal{A}$

Reminder: I: set of representatives of simple objects in $\mathcal{A}$.

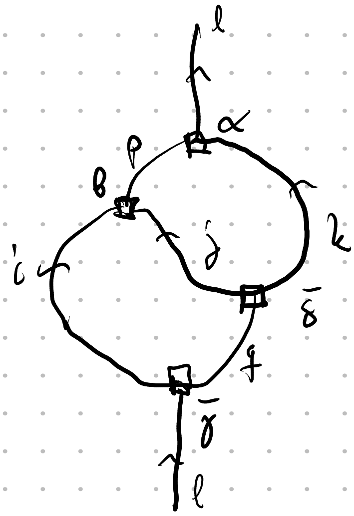
Fix bases $\Big\{$  $\Big\} \subset \mathrm{Hom}_{\mathcal{A}}(i \otimes j, k)$

$\Big\{$  $\Big\} \subset \mathrm{Hom}_{\mathcal{A}}(k, i \otimes j)$

which are dual:

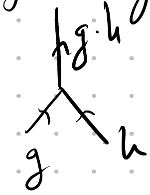
 $= \delta_{\alpha,\beta}$ 

F-matrices:

$\left(F^{l}_{ijk}\right)^{\gamma q \delta}_{\alpha p \beta} =$ 

Example:

- Vect_G: Simple objects: $g \in G$
  Fusion:



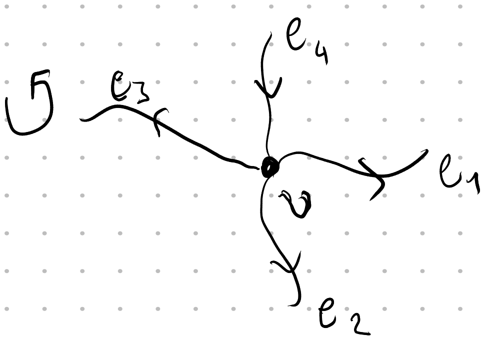
- $\mathrm{Rep}(G)$
- Fib: Simple obj: $\mathbb{1}, \tau$
  Fusion: $\tau \otimes \tau = \mathbb{1} \oplus \tau$

$$\left(F^{\tau}_{\tau\tau\tau}\right) = \frac{1}{\phi}\begin{pmatrix} 1 & \phi^{1/2} \\ \phi^{1/2} & -1 \end{pmatrix}$$

Surfaces w. Graphs

Let $\Gamma \subset \Sigma$ be an or. graph Σ. A coloring of Γ is a map $c: E(\Gamma) \to I$.
Let $v \in \Gamma$ be a vertex. ⇝ induces a cyclic ordered set $((e_1, \varepsilon_1), \dots, (e_n, \varepsilon_n))$ where
$\varepsilon_i = +$ if e_i is oriented outwards
$\varepsilon_i = -$ if e_i is oriented inwards.

e.g.



(opposite to Σ orientation)

$((e_1, +), (e_2, +), (e_3, +), (e_4, -))$

$$V(v,c) := \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, c(e_1)^{\varepsilon_1} \otimes \cdots \otimes c(e_n)^{\varepsilon_n}) \equiv \langle c(e_1)^{\varepsilon_1}, \ldots, c(e_n)^{\varepsilon_n} \rangle$$

where $X^+ := X$, $X^- := X^*$

Remark: The above depends on a choice of initial half-edge e_1. To get rid of this dependence;
Pivotality of $\mathcal{A}$ gives isos:

$$z_i : \langle c(e_i)^{\varepsilon_i}, \ldots, c(e_n)^{\varepsilon_n}, c(e_1)^{\varepsilon_1}, \ldots, c(e_{i-1})^{\varepsilon_{i-1}} \rangle$$
$$\cong \langle c(e_{i+1})^{\varepsilon_{i+2}}, \ldots, c(e_n)^{\varepsilon_n}, c(e_1)^{\varepsilon_1}, \ldots, c(e_i)^{\varepsilon_i} \rangle$$

which form a projective system.
Then, take $V(v,c) = \lim_{\text{cyclic}} \langle c(e_i)^{\varepsilon_i}, \ldots, c(e_{i-1})^{\varepsilon_{i-1}} \rangle$

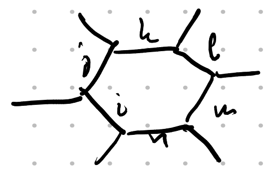
$$\rightsquigarrow V(c) = \bigotimes_{v \text{ vert.}} V(v,c)$$

$$\rightsquigarrow V(\Sigma; \Gamma) = \bigoplus_{c \text{ coloring}} V(c)$$

$H(\Sigma) = \mathbb{C}[\mathcal{A}\text{-colored graphs in } \Sigma]$

$\downarrow$

$SN(\Sigma) = H(\Sigma) / \text{local relations}$

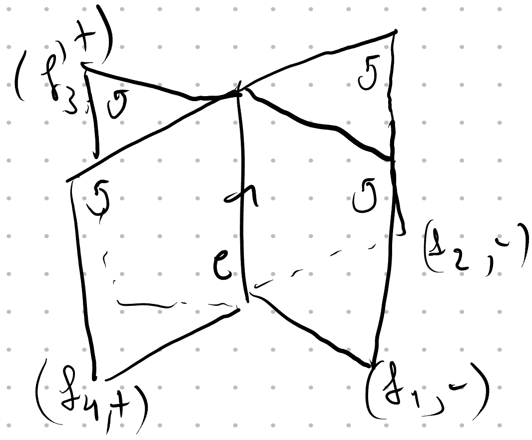


3D picture [T, Vir]

Let M be a closed 3-d mfld w. $P \subset M$ a skeleton of M.
A coloring of P is a map $c : F(P) \to I$

· The orientation of e and M induce a cyclic order $((f_1,\varepsilon_1),\dots,(f_n,\varepsilon_n))$ of incident faces f_i and $\varepsilon_i\in\{\pm\}$ determined by the or. of f_i.

E.g.:

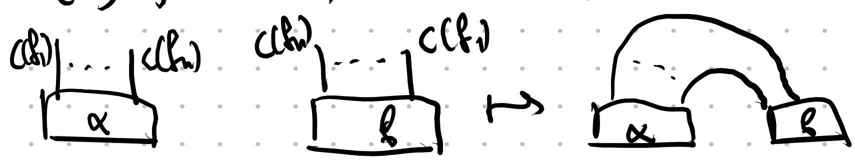


$$\rightsquigarrow \quad V(\vec{e},c) = \mathrm{Hom}(\mathbb{1}, c(f_1)^{\varepsilon_1}\otimes\cdots\otimes c(f_n)^{\varepsilon_n})$$

$$V(\overleftarrow{e},c) = \mathrm{Hom}(\mathbb{1}, c(f_n)^{-\varepsilon_n}\otimes\cdots\otimes c(f_1)^{-\varepsilon_1})$$

$$\rightsquigarrow \quad V(c) := \bigotimes_{\vec{e}\text{ orient.}} V(\vec{e},c)$$

$V(\vec{e},c)\otimes V(\overleftarrow{e},c)\longrightarrow \mathbb{k}$ non-deg.



$\rightsquigarrow$ Copairing: $\mathbb{k}\to V(\overleftarrow{e},c)\otimes V(\vec{e},c)$

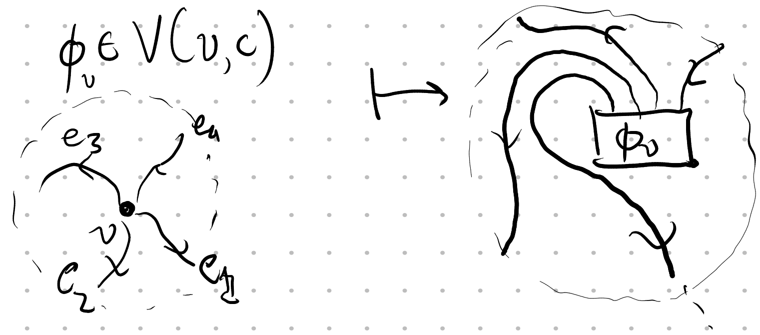
$$\longrightarrow \quad \star_e : V(\vec{e},c)^*\otimes V(\overleftarrow{e},c)^*\longrightarrow \mathbb{k}.$$

$$\Rightarrow \left[\star_c = \bigotimes_{e\text{ unori.}} \star_e : V(c)^*\longrightarrow \mathbb{k}\right]$$

Let $\Gamma \subset \mathbb{R}^2$, then $\exists$ lin map

$$\mathbb{F}(\Gamma, c) : V(\mathbb{R}^2; \Gamma, c) \to \Bbbk$$

- $\phi_v \in V(v,c)$

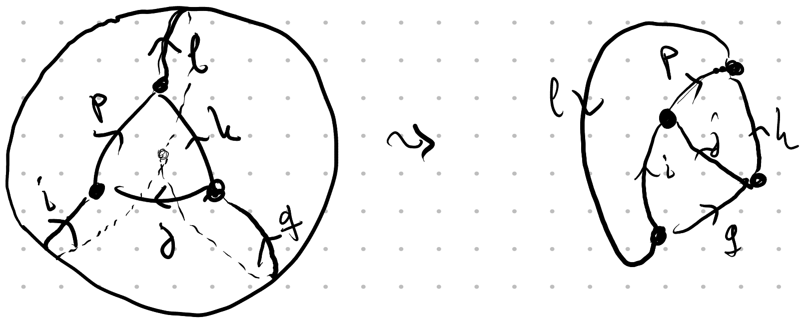


This extends to

$$\mathbb{F}(\Gamma, c) : V(S^2, \Gamma, c) \to \Bbbk.$$

(by sphericality)

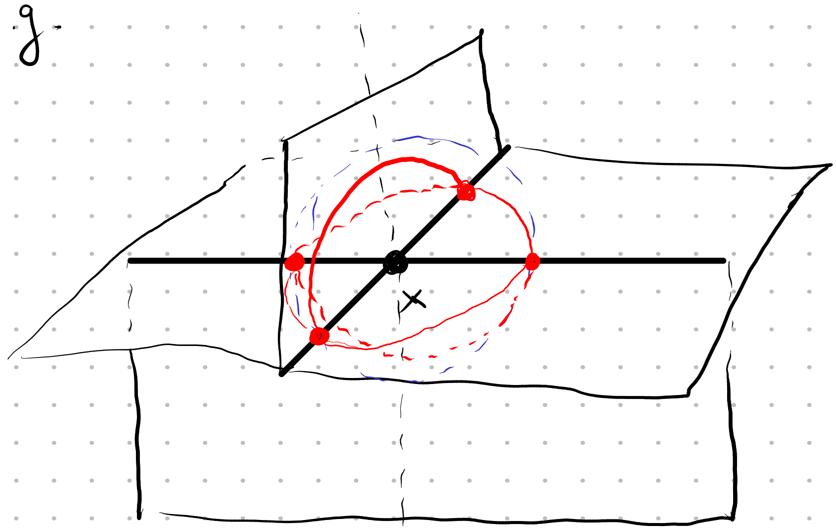
<u>Example:</u>



⇝ 6j-symbols (F-matrices)

Let $x \in P$ be a vertex. Consider B_x a small 3-ball around x. The intersection $\partial B_x \cap P = \Gamma_x$ is a graph on ∂B_x.

E.g.



$$\mathbb{F}(\Gamma_x, c) \in V(\partial B_x; \Gamma_x, c)^*$$

Notice

$$V(\partial B_x, \Gamma_x, c) = \bigotimes_{\vec{e}_x} V(\vec{e}_x, c)$$

$\vec{e}_x$ incident half-edges or. outwards

Turaev-Viro Invariants:

$$Z^{TV}(M) = (\dim(A))^{-|M \setminus P|} \cdot \sum_{c \text{ coloring}} \prod_{f \text{ faces}} \dim c(f)^{\chi(f)} \ast_c \left(\bigotimes_x F(\Gamma_x, c)\right)$$

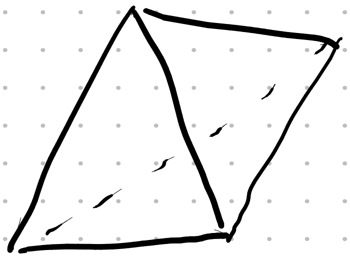
$|M \setminus P|$ ← number of 3-strata

$$\dim(A) = \sum_{i \in I} \dim(i)^2$$

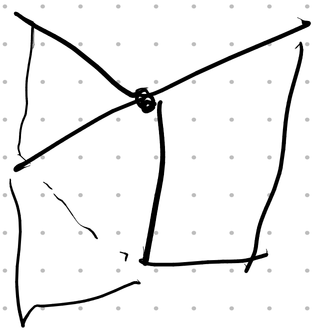
$$Z^{TV}(S^3) = \dim(A)^{-2} \sum_{i \in I} \dim(i)^2 = \dim(A)^{-1}$$

$S^2 \subset S^3$

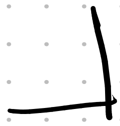
Rem: Original



dual



1-simpl.: $i \in I$
2-simpl: sp. of admissible labels
3-simpl: 6j-symbols.



Rem:
TV can be extend to M w. $\partial M \neq \emptyset$

$(\Sigma \times I, P) : (\Sigma, \Gamma) \to (\Sigma, \Gamma') \rightsquigarrow Z(\Sigma \times I, P) : V(\Sigma, \Gamma) \to V(\Sigma, \Gamma')$

$\rightsquigarrow V^{TV}(\Sigma) := \lim_{\Gamma} V(\Sigma, \Gamma)$

Relation to String Net Models [Kir,]

$$SN(\Sigma) = \mathbb{C}[A\text{-label graphs in } \Sigma] / \text{local relation.}$$

$\exists \; V(\Sigma, \Gamma) \longrightarrow SN(\Sigma)$

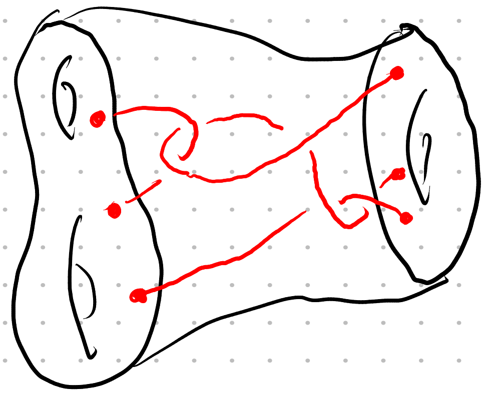
Thm: (Kir) $V^{TV}(\Sigma) \cong SN(\Sigma)$

Rem:
- [Bartlett, Goosen] $\rightsquigarrow$ TV $\cong$ SN as 321-TQFTs
- Boundary is described by $Z(A)$

- Recall: $\mathcal{C}$ Modular fusion Cat
  ⇝ Reshetikhin-Turaev 3 TQFT $Z^{RT,\mathcal{C}}$

[T, Vir]: $Z^{TV,A} \cong Z^{RT,Z(A)}$

→ Wilson lines can be included



Wilson lines are colored by objects of $\underline{Z(A)}$.

2) <u>Top. Quantum Computing</u>

V: surf. w. punctures → Vect

(↗ Modular functor)

In particular, braid grp reps.

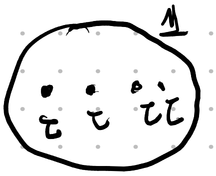
Recall: A gate set S (subset of $\bigcup_n U(2^n)$) is <u>universal</u> if {eigquantum circuits over S} is dense in $SU(2^n)$. $\forall n$.

TQFT (Mod. functor) → Input info in Quantum Sys. → Process via syst. dynamics (braid group reps)

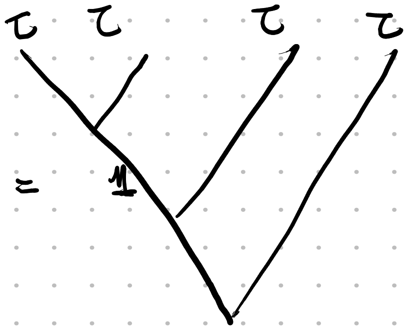
To encode qubits in V, find an embedding

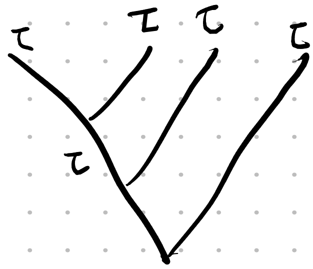
$$(\mathbb{C}^2)^{\otimes n} \hookrightarrow V$$

Example: • Fib

$V_4^1 = V\Big($  $\Big) = \mathrm{Hom}(\mathbb{1}, \tau^{\otimes 4}) \cong \mathbb{C}^2$

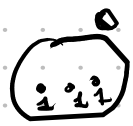
basis

$|0\rangle =$ 

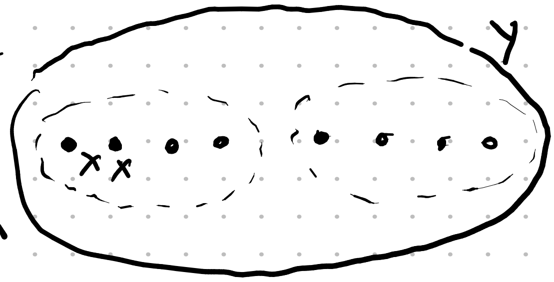
$|1\rangle =$ 

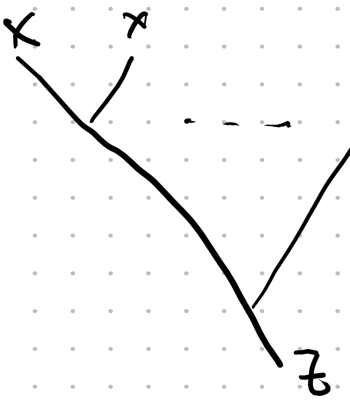
• SU(2)-Chern Simons theory at $k=3$

$I = \{0,1,2,3\}$

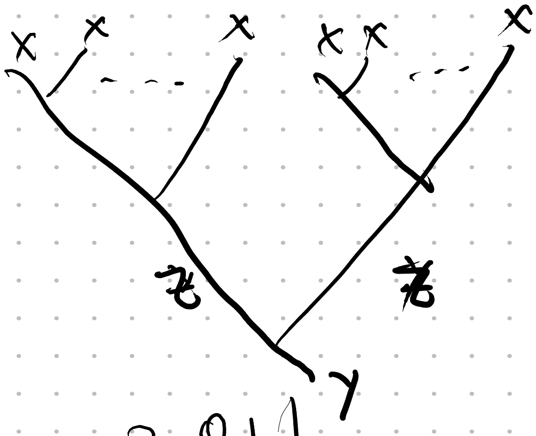
$\rightsquigarrow V_3^1 = V\Big($  $\Big) \cong \mathbb{C}^2$ [FLW]

To encode n-qubits, use the gluing for

$V\Big($  $\Big) \cong \bigoplus_{z,z'} V\Big($  $\Big) \otimes V\Big($  $\Big) \otimes V\Big($  $\Big)$



Qubit



2-Qubit.

⇝ This is called <u>sparse</u> encoding

<u>Rem:</u> Often, there is leakage (braids do not preserve the n-qubit subsp.)

<u>Universality:</u> implement universal gates using braidings.

Roughly: find braid b

$$\begin{array}{ccc} (\mathbb{C}^2)^{\otimes n} & \hookrightarrow & V_n \\ U\downarrow & & \downarrow V(b) \\ (\mathbb{C}^2)^{\otimes n} & \hookrightarrow & V_n \end{array}$$

related to B_n-reps having dense image.

<u>Example:</u>

- Fib allows universal computation.
- $SU(2)_3$-CS theory —"— [FLW]
- Kitaev's toric is not universal

↑

$\mathcal{A} = \mathrm{Vect}_{\mathbb{Z}_2}$ ⇝ Vect_G

Rem: (besides braidings)

- Use ancillary top. states / measurements to do universal comp.

  [Mochon; 0306063] $G = S_3$

- Use the whole MCG reps (on higher genus) w/ defects [Barkeshli, Freedman]

Rem: Simulate TQFTs

$$\begin{array}{ccc} V(\Sigma) & \hookrightarrow & (\mathbb{C}^2)^{\otimes n} \\ V(f) \downarrow & & \downarrow U \\ V(\Sigma) & \hookrightarrow & (\mathbb{C}^2)^{\otimes n} \end{array}$$