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ZMP seminar:

The toric code (a.k.a. Kitaev model)

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The toric code is a simple example of a quantum many-body system in two dimensions, demonstrating the characteristic properties of topological phases of matter.

1. Hilbert space and Hamiltonian

Basic building block: $H := \mathbb{C}^2$

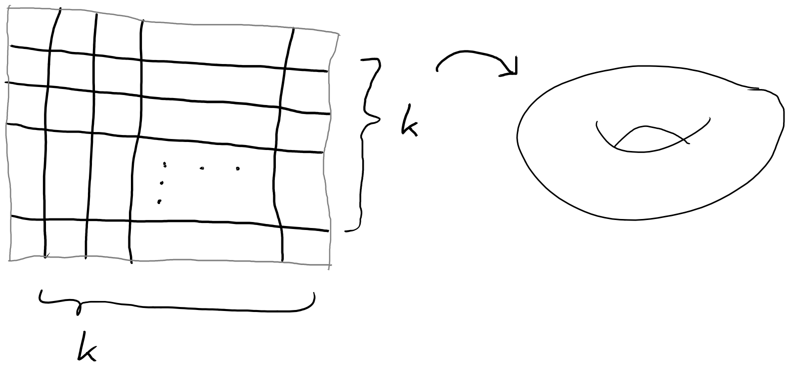
with orth. norm. basis $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ "spin up"

$|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ "spin down"

Operators: $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^2) = \mathrm{span}_{\mathbb{C}}\{ id, X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$ $Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \}$ with relations

$X \cdot Z = iY = -Z \cdot X \qquad X^2 = id = Z^2.$

Consider a square lattice of length k on the torus:



V: set of vertices

E: —"— edges

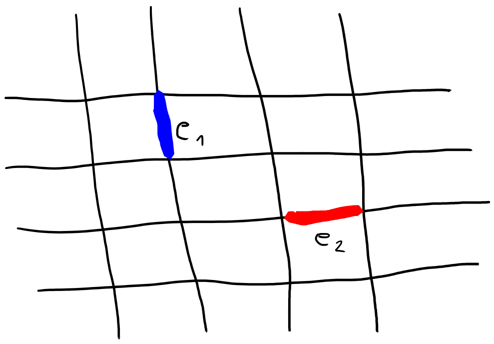
F: —"— faces (a.k.a. plaquettes)

Def. Hilbert space of the toric code: $\mathcal{H} := \bigotimes_{e\in E} \mathbb{C}^2$.

Orth. norm. basis $\left\{ \bigotimes_{e\in E} |i_e\rangle \;\middle|\; i : E \to \{0,1\} \right\}$.

Operator algebra $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$ is generated by: X_e, Z_e for all $e \in E$.

Picture example:



$:= \underline{X_{e_1}}\, \underline{Z_{e_2}}$

We write for any subset $S \subseteq E$,

$X_S := \prod_{e\in S} X_e$ and $Z_S := \prod_{e\in S} Z_e$,

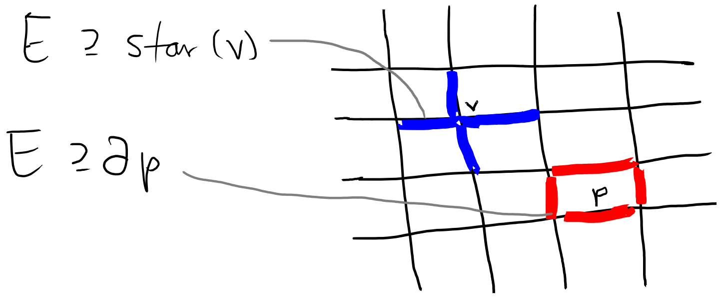
and call S the *support* of X_S or Z_S, resp.

For $v \in V$ consider

$$A_v := X_{\mathrm{star}(v)} = \prod_{e \in \mathrm{star}(v)} X_e \,, \qquad \text{vertex operator}$$

and for $p \in F$

$$B_p := Z_{\partial p} = \prod_{e \in \partial p} Z_e \,, \qquad \text{plaquette operator}$$



A_v and B_p are hermitian operators with eigenvalues 1 and -1,

$A_v^2 = \mathrm{id}$ $\quad B_p^2 = \mathrm{id}$,

and they commute $\forall v \in V, p \in F$.

<u>Def.</u> Hamiltonian on $\mathcal{H}$:

$$h := -\sum_{v \in V} A_v - \sum_{p \in F} B_p$$

Eigenspaces are determined by all possible combinations of signs:

$A_v |\xi\rangle = \pm |\xi\rangle$, $B_p |\xi\rangle = \pm |\xi\rangle$ $\quad \forall v \in V, p \in F$.

2. Ground-state space

$$\mathcal{H}_0 := \left\{ |\xi\rangle \in \mathcal{H} \;\middle|\; \begin{array}{l} A_v|\xi\rangle = |\xi\rangle \quad \forall v \in V, \\ B_p|\xi\rangle = |\xi\rangle \quad \forall p \in F \end{array} \right\} \subseteq \mathcal{H}$$

We determine $\mathcal{H}_0$ by analyzing its operator algebra $\mathrm{End}_{\mathbb{C}}(\mathcal{H}_0) \cong \mathcal{G} / \langle A_v - \mathrm{id}, B_p - \mathrm{id} \rangle$, where

$$\mathcal{G} := \{ T \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}) \mid T(\mathcal{H}_0) \subseteq \mathcal{H}_0 \} \subseteq \mathrm{End}_{\mathbb{C}}(\mathcal{H}).$$

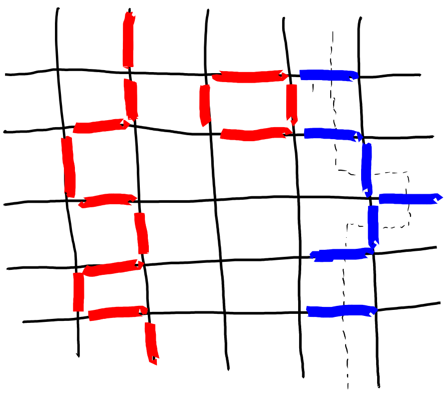
Recall: $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$ is generated by

$$\{ X_S, Z_S \mid S \subseteq E \}.$$

<u>Lemma</u> $\mathcal{G} = \{ T \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}) \mid T(\mathcal{H}_0) \subseteq \mathcal{H}_0 \}$

is generated by

$$\left\{ X_c, Z_{c'} \;\middle|\; \begin{array}{l} c \text{ closed path in the graph } (V,E) \\ c' \text{ ——— '' ——— in the dual graph } (F,E) \end{array} \right\}$$



In red: two closed paths in (V,E)

In blue: a closed path in the dual graph (F,E)

Proof For $T : \mathcal{H} \to \mathcal{H}$,

$$T(\mathcal{H}_0) \subseteq \mathcal{H}_0 \text{ iff } \begin{cases} TA_v = A_v T & \forall v \in V \\ TB_p = B_p T & \forall p \in F \end{cases}.$$

For $S, S' \subseteq E$ consider Z_S and $X_{S'}$.

Recall that $A_v = X_{star(v)}$ and $B_p = Z_{\partial p}$.

We always have $A_v X_{S'} = X_{S'} A_v \quad \forall v \in V$

and $B_p Z_S = Z_S B_p \quad \forall p \in F$.

However, $A_v Z_S = (-1)^{|S \cap star(v)|} Z_S A_v$

and $B_p X_{S'} = (-1)^{|S' \cap \partial p|} X_{S'} B_p$.

Now, $|S \cap star(v)|$ is even $\forall v \in V$

iff $S \subseteq E$ is a disjoint union of closed paths in (V, E).

Similarly, $|S' \cap \partial p|$ is even $\forall p \in F$

iff $S' \subseteq E$ is a disjoint union of closed paths in the dual graph (F, E).

$\square$

Next observation

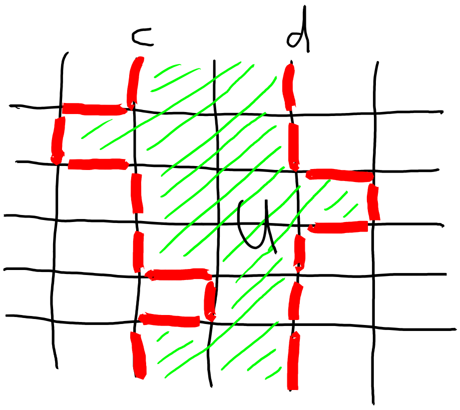
$$Z_{c_1} = Z_{c_2} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}_0) \iff$$

c and d are homotopic closed paths in (V,E).

$$X_{c_1'} = X_{c_2'} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}_0) \iff$$

c' and d' are homotopic closed paths in dual graph (F,E).

Indeed, if $U \subseteq F$ such that $\partial U = c \cup d$, then

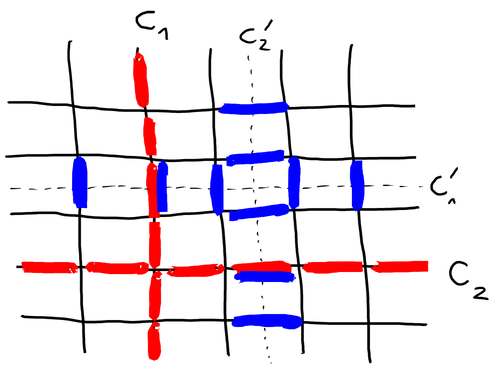


$$Z_c \cdot Z_d = \prod_{p \in U} B_p = \mathrm{id}_{\mathcal{H}_0}$$

in $\mathrm{End}_{\mathbb{C}}(\mathcal{H}_0)$.

The statement about $X_{c'}$, $X_{d'}$ is completely analogous.

Upshot Let c_1, c_2 and c_1', c_2' be representative, non-homotopic cycles in (E,V) and its dual (F,V), respectively.



Then $\mathrm{End}_{\mathbb{C}}(\mathcal{H}_0)$ is generated by $\{Z_{c_1}, X_{c_1'}, Z_{c_2}, X_{c_2'}\}$

Relations: $Z_i X_i = -X_i Z_i$, $Z_i^2 = \mathrm{id} = X_i^2$

$\Rightarrow \mathcal{H}_0 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$. $\rightsquigarrow$ Degenerate ground-state space.

For norm. $|\xi\rangle \in \mathcal{H}_0$ s.t. $Z_i|\xi\rangle = |\xi\rangle$

$\rightsquigarrow$ ON basis $\{\xi, X_1\xi, X_2\xi, X_1X_2\xi\}$.

Remark This construction generalizes to any compact oriented surface Σ and finite group G. (here: $G = \mathbb{Z}_2$)

Then $\mathcal{H}_0 \cong \mathbb{C}\,\mathrm{Hom}_{Grp}(\pi_1(\Sigma), G)/G$.

3. The toric code as an error-correcting quantum code

In a *quantum code*, information is stored as a state in a subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ of a quantum system.

An *error* is a linear operator $E : \mathcal{H} \to \mathcal{H}$.

In the toric code, we can detect errors by measuring A_v and B_p.

However, this does not detect errors s.t. $E(\mathcal{H}_0) \subseteq \mathcal{H}_0$.

On the other hand, as we saw, for this, E must have non-contractible cycles in its support.

We can thus say that it is possible to detect any error of size $|\mathrm{supp}(E)| \leq k$.

Stability under local perturbations

Add to the Hamiltonian $h = -\sum_v A_v - \sum_p B_p$ a perturbation V that involves only terms of at most two spins $e \in E$ interacting. Let $|\xi\rangle, |\eta\rangle \in \mathcal{H}_0$.

According to standard perturbation theory, the energy splitting of $|\xi\rangle$ and $|\eta\rangle$ in m-th order is proportional to $\langle \xi | V^m | \eta \rangle$ or $\langle \xi | V^m | \xi \rangle - \langle \eta | V^m | \eta \rangle$.

But these can be non-zero only if V^m involves interactions of at least k spins (non-contractible loop).

4. Local excitations (quasi-particles)

An *elementary excitation* is a state $|\Psi\rangle \in \mathcal{H}$ s.t. $A_v|\Psi\rangle = |\Psi\rangle$ and $B_p|\Psi\rangle = |\Psi\rangle$ for all $v \in V$, $p \in F$ except exactly one $v' \in V$ or $p' \in F$.

If $A_{v'}|\Psi\rangle \neq |\Psi\rangle$, then *type Z*.

If $B_{p'}|\Psi\rangle \neq |\Psi\rangle$, then *type X*.

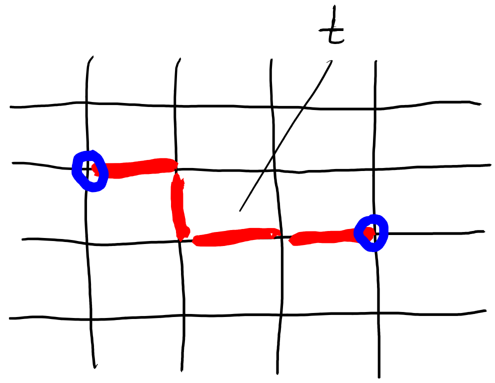
Observation: A single excitation cannot exist, because $\prod_{v \in V} A_v = id_{\mathcal{H}}$ and $\prod_{p \in F} B_p = id_{\mathcal{H}}$. For example, if $A_v|\Psi\rangle = |\Psi\rangle$ $\forall v \in V \setminus \{v'\}$, then

$$A_{v'}|\Psi\rangle = \prod_{v \in V} A_v |\Psi\rangle = |\Psi\rangle .$$

However, pairs of excitations do exist:

Consider a non-closed path $t \subseteq E$ in the graph (V, E).

Then for any $|\xi\rangle \in \mathcal{H}_0$,



the state $Z_t |\xi\rangle \in \mathcal{H}$ has two type-z excitations at the endpoints $v_1, v_2 \in V$ of t.

<u>Why</u>? $\quad B_p Z_t |\xi\rangle = Z_t |\xi\rangle \quad \forall p \in F.$

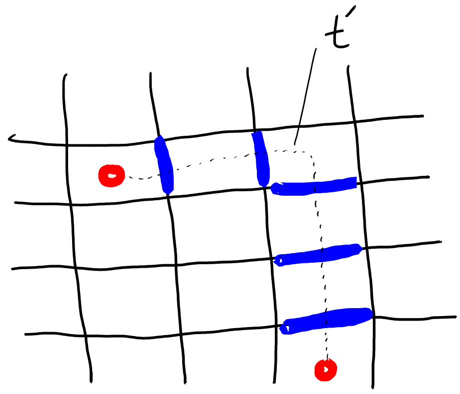
For $v \neq v_1, v_2$ t shares evenly many edges with star(v) $\Rightarrow A_v Z_t |\xi\rangle = Z_t |\xi\rangle$.

The stars of $v_1, v_2 \in V$ share an odd number of edges with $t \subseteq E$.

$$\Rightarrow A_{v_1} Z_t |\xi\rangle = - Z_t |\xi\rangle$$

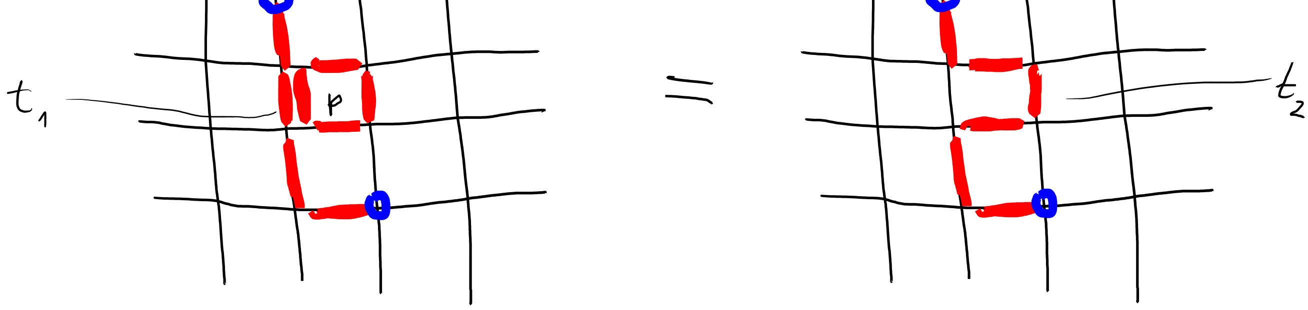
and $A_{v_2} Z_t |\xi\rangle = - Z_t |\xi\rangle$.

Dually, for $|\xi\rangle \in \mathcal{H}_0$ and $t' \subseteq E$ a non-closed path in (F, V), $X_{t'}|\xi\rangle \in \mathcal{H}$ has two x-type excitations at the endpoints $p_1, p_2 \in F$ of t'.



<u>Observation</u> The states $Z_t|\xi\rangle$ and $X_{t'}|\xi\rangle$ depend on the paths t and t' only up to homotopy fixing the endpoints.

<u>Example</u>

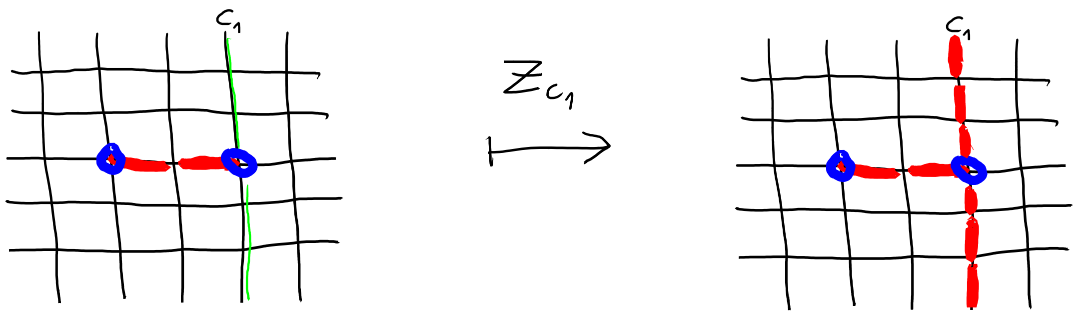


$$\Rightarrow Z_{t_2}|\xi\rangle = Z_{t_1} B_p |\xi\rangle = Z_{t_1}|\xi\rangle \quad \text{for } |\xi\rangle \in \mathcal{H}_0 .$$

In this way, one can create states with any configuration of even numbers of x-type particles and of z-type particles.

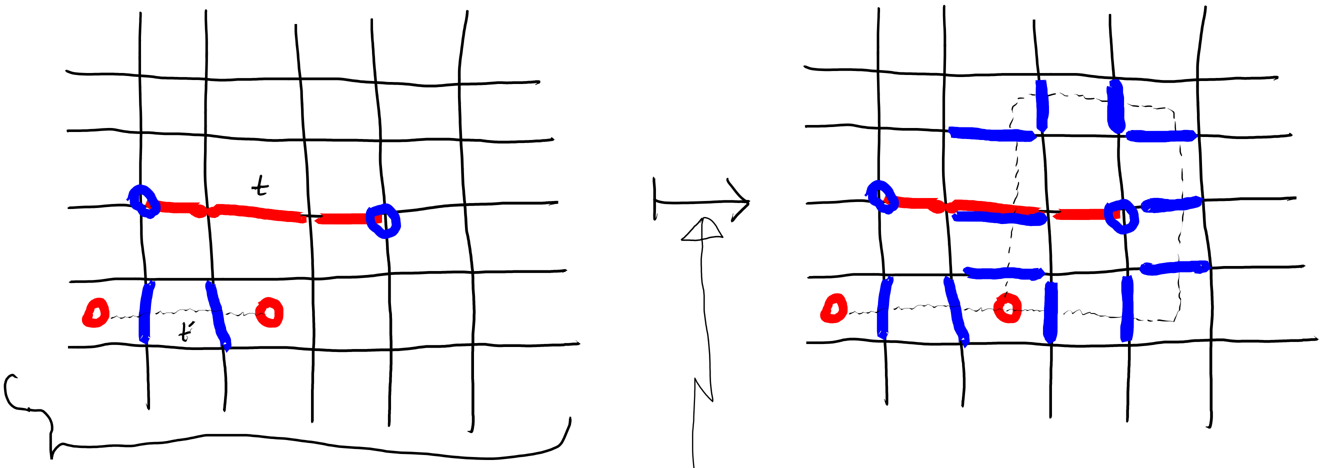
5. The quasi-particles are anyons

The excitations can be moved around in the surface by the operators $X_{t'}$, Z_t. For example:



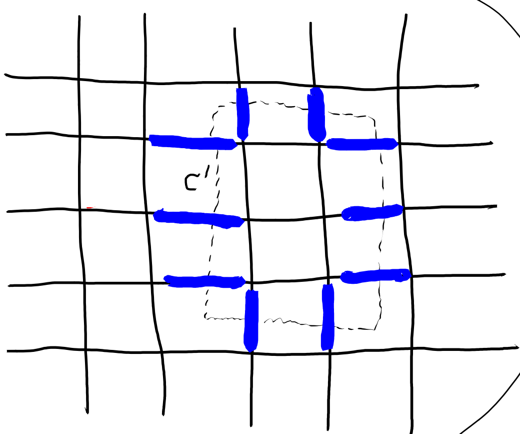
Here, one z-type particle moves around the torus along a non-contractible cycle.

We want to demonstrate that the quasi-particles in the toric code are anyons, i.e. one obtains non-trivial operators from braiding particles around each other. E.g. Let $|\xi\rangle \in \mathcal{H}_0$.



$Z_t X_{t'} |\xi\rangle$

$X_{c'} =$

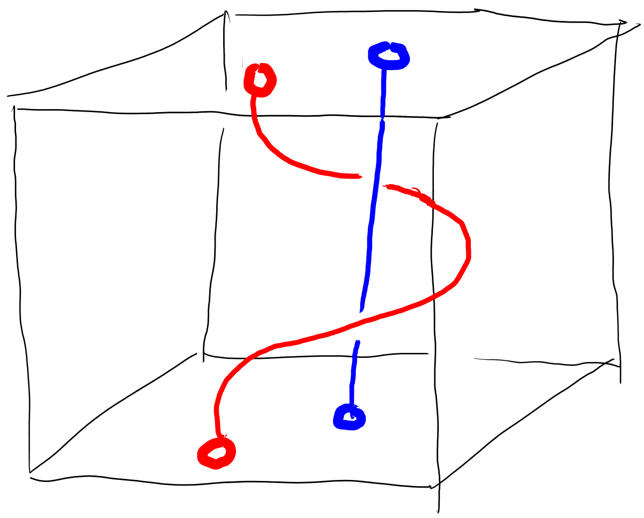


We compute: $X_{c'} Z_t X_{t'} |\xi\rangle$

$= - Z_t X_{t'} X_{c'} |\xi\rangle = - Z_t X_{t'} |\xi\rangle$.

Outlook

In the underlying topological field theory, such an operator is assigned to a 3-manifold with embedded braids:



For topological quantum computing, such operators are used to implement quantum gates, elementary operations on the quantum code $\mathcal{H}_0 \subseteq \mathcal{H}$.