Levin-Wen Models

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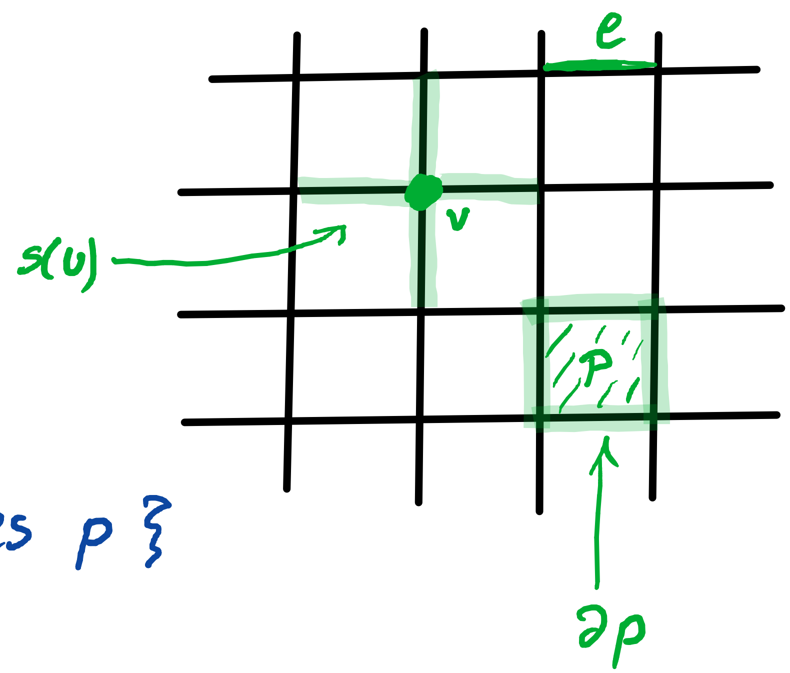
ZMP Seminar 11.06.20

[Levin, Wen: cond-mat/0404617]

• Recollection on the Kitaev model (Vincent)

- We considered a square lattice with periodic boundary conditions, i.e. on a torus

- Notation: $V = \{\text{vertices } v\}$



$E = \{\text{edges } e\}$

$F = \{\text{plaquettes / faces } p\}$

$s(v)$ = star of v

∂p = boundary of p

- Each edge carries a Hilbert space $\mathbb{C}^2$, i.e. a qbit
- We considered the operators

$$\mathbb{1} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad X = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad Y = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

flips a qbit

measures "up" vs "down"

- For $S \subseteq E$, we defined $X_S := \prod_{e \in S} X_e$, $Z_S := \prod_{e \in S} Z_e$

as well as $A_v := X_{star(v)}$, $B_p := Z_{\partial p}$.

- Hilbert space: $\mathcal{H} = (\mathbb{C}^2)^{\otimes E}$

- Hamiltonian: $H = -\sum_{v\in V} A_v - \sum_{p\in F} B_p$

- Key observation: for all $v, w \in V$, $p, q \in F$,

$$[A_v, A_w] = 0, \quad [B_p, B_q] = 0, \quad [A_v, B_p] = 0$$

$\Rightarrow$ All terms in H commute

$$\Rightarrow \mathcal{H}_0 = \left\{ |\psi\rangle \in \mathcal{H} \;\middle|\; \begin{array}{l} A_v|\psi\rangle = |\psi\rangle, \; B_p|\psi\rangle = |\psi\rangle \\ \forall v \in V, p \in P \end{array} \right\}$$

• Recollection on Quantum Computing

- We found that the elementary excitations of Kitaev's toric code were anyonic.

- Braiding anyons gives a way to implement quantum gates, $B_N \longrightarrow \mathcal{U}(\mathcal{H})$.

- Quantum computer is universal if this map has dense image.

⇝ Kitaev's toric code is not universal!

⟹ Find generalisations!

• Spherical Fusion Categories

• $\mathcal{C}$ is :

$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$

- monoidal $\rightsquigarrow$ $(\otimes, a, \mathbb{1}, \lambda, \rho)$
- $\mathbb{C}$-linear $\rightsquigarrow$ $\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Vec}$ is $\mathbb{C}$-vsp.
- abelian + semi-simple $\rightsquigarrow$ Direct-sum decomposition into objects $X \in \mathcal{C}$ with $\mathrm{Hom}_{\mathcal{C}}(X,X) \cong \mathbb{C}$

↳ These are called <u>simple</u>

- $I = \{X \in \mathcal{C} \mid X \text{ simple}\}/\text{iso}$ is finite
- $\mathbb{1}$ is simple

- rigid $\rightsquigarrow$ every $x \in \mathcal{C}$ has both duals : ${}^{\vee}x$, $x^{\vee}$
- pivotal $\rightsquigarrow$ monoidal natural iso $p_x : {}^{\vee}x \xrightarrow{\cong} x^{\vee}$

  $\Rightarrow$ induces $(x^{\vee})^{\vee} \cong x$.

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- E.g.: Representation cats of finite groups with the tensor product of representations as monoidal structure.

- Fix a representative $X_i \in \mathcal{C}$ in each iso-class of simples.

- Fusion coefficients: $N_{ij}^k := \dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_k)\right)$

$$\Rightarrow \quad X_i \otimes X_j \cong \bigoplus_{k \in I} N_{ij}^k \cdot X_k .$$

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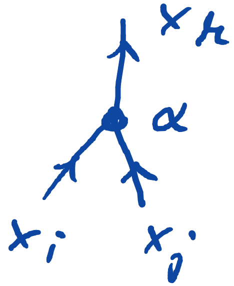
- F-matrices / F-symbols:

choose + fix bases

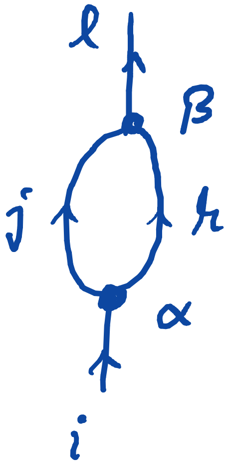
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}((x_i \otimes x_j) \otimes x_k, x_l) \arrow[d, "\cong" ', "\text{from } a_{ijk}"] \arrow[r, phantom, "\cong"] & \sum_{r \in I} N_{ij}^r N_{rk}^l \cdot \mathbb{C} \arrow[d, bend left, "\cong" ', "F_{ijk}^l"] \\
\mathrm{Hom}_{\mathcal{C}}(x_i \otimes (x_j \otimes x_k), x_l) \arrow[r, phantom, "\cong"] & \sum_{s \in I} N_{is}^l N_{jk}^s \cdot \mathbb{C}
\end{tikzcd}$$

Crucial relations:

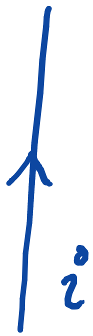


= basis element $(f_{ij}^{k})^{\alpha}$ in $\mathrm{Hom}_{\mathcal{C}}(x_i \otimes x_j, x_k)$

(1)



$= \delta_{\alpha\beta} \; \delta_{il}$



basis + dual basis

$\mathrm{Hom}_{\mathcal{C}}(x_i, x_l) \cong \delta_{il} \cdot \mathbb{C}$.

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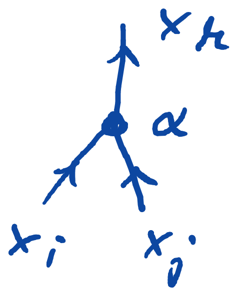

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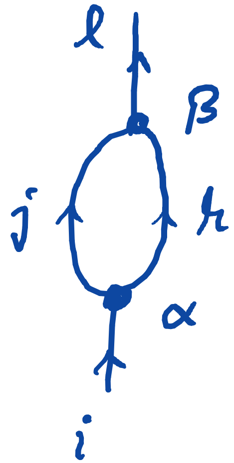

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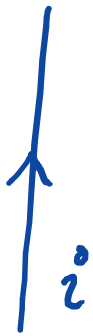
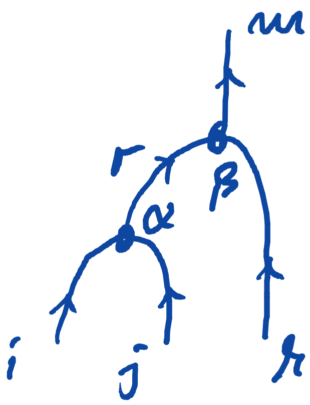


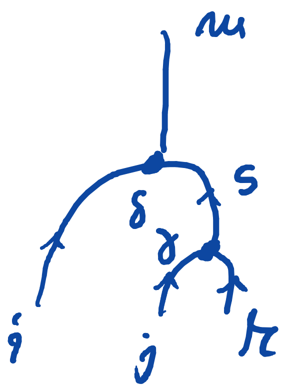
basis + dual basis

$\mathrm{Hom}_{\mathcal{C}}(x_i, x_l) \cong \delta_{il} \cdot \mathbb{C}$.

(2)



$= \sum_{s\gamma\delta} \left(F^{m}_{ijk}\right)^{s\gamma\delta}_{r\alpha\beta}$



⇝ pentagon identity for Fs!

· Levin-Wen Models (String-Net Models)

- We could view the toric code as a $\mathbb{Z}_2$ lattice gauge theory

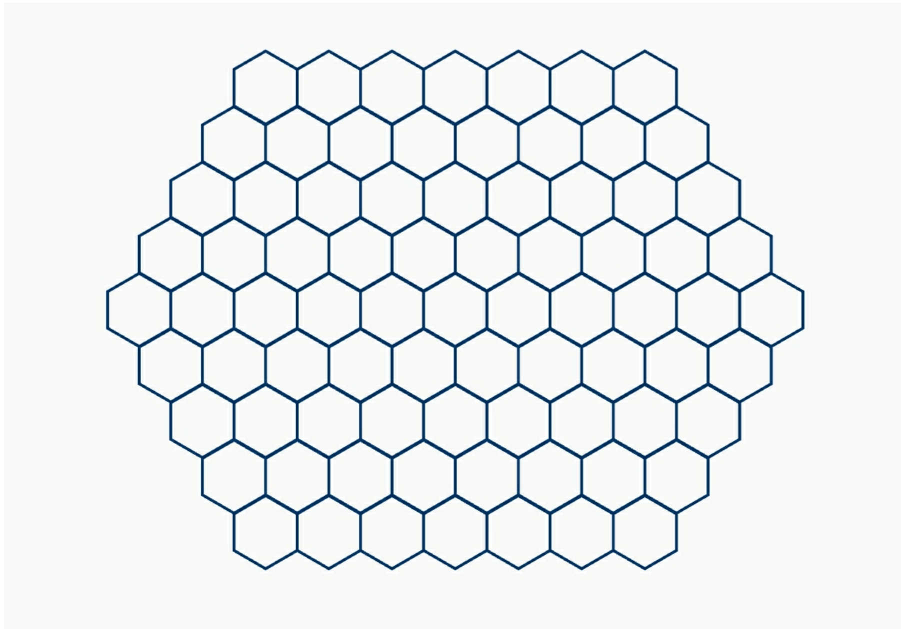
↳ Can generalise this to other groups G

↳ Edges labelled by irreducible representations of G

↳ Generalise : Label edges by simple objects $\{X_i\}_{i \in I}$ in a spherical fusion category $\mathcal{C}$.

• Set-up and States

• Work on trivalent lattice (for convenience)



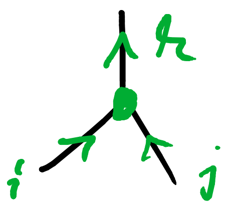
• Edges are endowed with

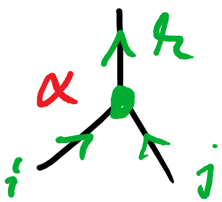
  • orientation
  • Labels $i \in I$

$$\left.\begin{array}{l}\cdot\ \text{orientation}\\ \cdot\ \text{Labels } i\in I\end{array}\right\}\quad (e, i) \updownarrow (\bar{e}, i^{\vee})$$

• We choose a basis $(f_{i,j}^{k})_{\alpha}$ of $\mathrm{Hom}_{\mathcal{C}}(x_i \otimes x_j, x_k)$ for all $i, j, k \in I$.

• Vertices are endowed with a basis element





• Hilbert Space

• We work with a finite, periodic lattice.

• Then,

$$\mathcal{H} = \mathbb{C}[\mathcal{C}\text{-labelled lattice configurations}]$$

$$\cong \underbrace{\mathrm{Map}(\underbrace{\mathcal{C}\text{-labelled lattice configurations}}_{\text{classical states = lattice configs.}}, \mathbb{C})}_{\text{"wave functions"}}$$

- Interpretation: $e \in E$ labelled by $i \in I$

  - Let $0 \in I$ correspond to the simple obj. $x_0 = \mathbb{1}$.

↳ Then:
  - $i = 0$: e is unoccupied.
  - $i \neq 0$: e is occupied by a string of type i.

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↳ Then:
  - $i = 0$: e is unoccupied.
  - $i \neq 0$: e is occupied by a string of type i.

- E.g.: $\mathcal{C} = (\mathrm{Rep}_{\mathbb{Z}_2}, \otimes)$ ⇝ simples: $\underset{\displaystyle 1}{\overset{\mathbb{1}}{\shortparallel}}$, $\underset{\displaystyle -1}{\overset{\tau}{\shortparallel}}$

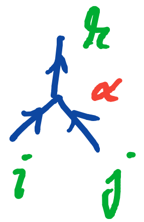
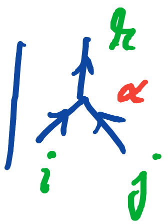
⇝ The Levin-Wen Hilbert space agrees with that of the toric code on the honeycomb lattice.

- ## Hamiltonian

- **Idea:** Follow the philosophy that LW models generalise the toric code.

$$H = -\sum_{v\in V} Q_v - \sum_{p\in F} B_p \qquad \text{(compare Kitaev model)}$$

- **Q_v:** "measures electric charges", favours no charge at v,

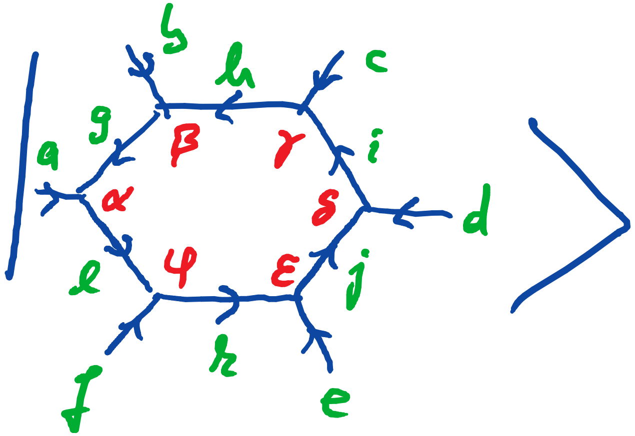
$Q_v \Big|$  $\Big\rangle = \delta_{ij}^{k} \Big|$  $\Big\rangle$,

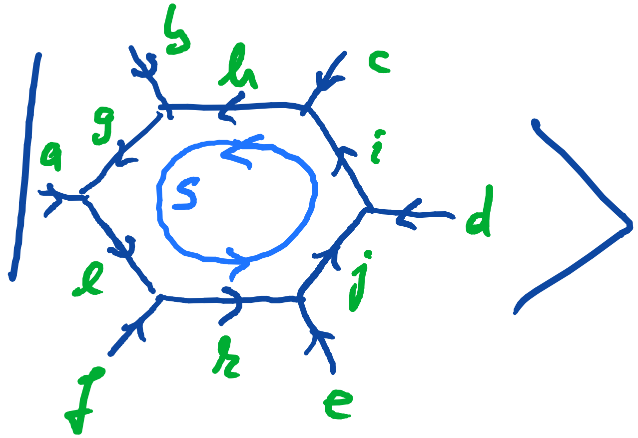
$$\delta_{ij}^{k} := \begin{cases} 1, & N_{ij}^{k} \neq 0 \\ 0, & N_{ij}^{k} = 0 \end{cases}$$

$\rightsquigarrow$ Favours non-vanishing fusion coefficients.

- B_p : "measures magnetic flux around p", favours no flux.

  - Need to measure all types of flux : $B_p = \sum_{s \in I} a_s \cdot B_p^s$.

  - Action: B_p^s 

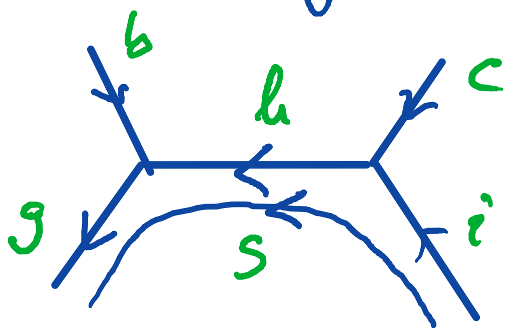
= 

    ⇝ B_p^s inserts a type-s closed string into p.

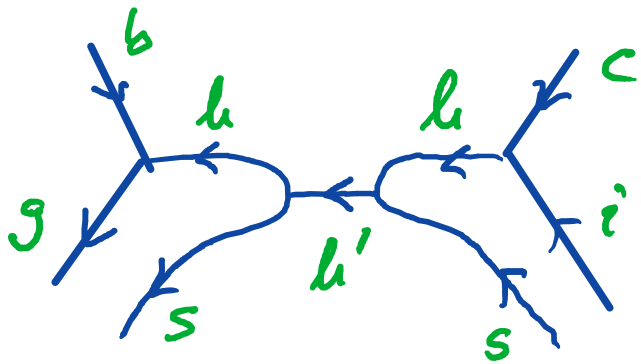
    ⇝ Have to make this into a lattice state again!

- Idea: Use $\mathcal{C}$'s string diagram calculus locally.

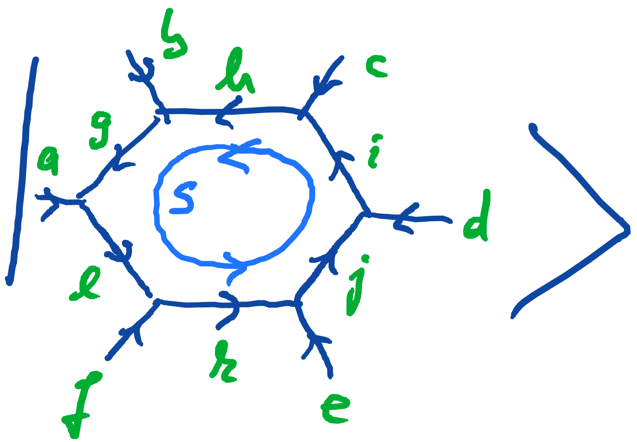
• Omitting basis indices, the idea of the computation is:



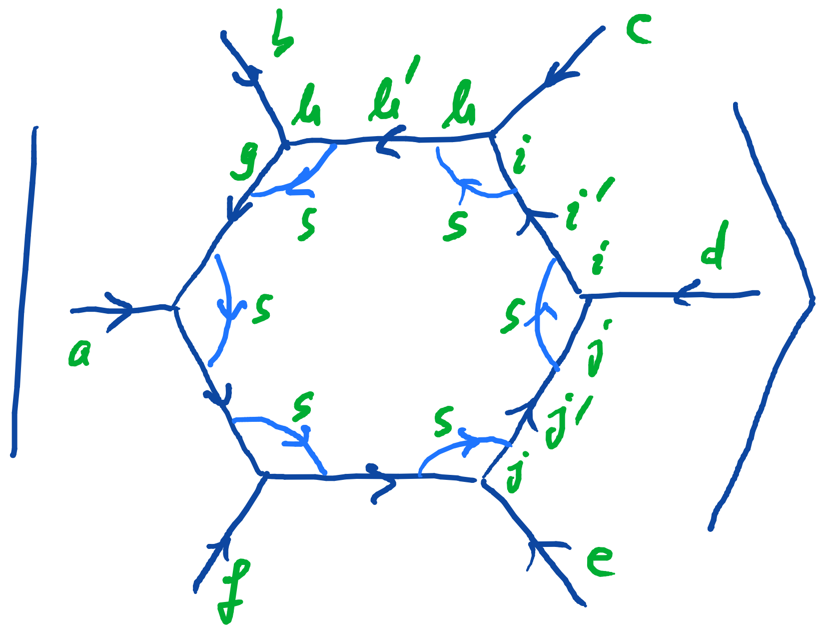
$$= \sum_{h'}$$



$$\Rightarrow$$



$$= \sum_{i', h', \dots}$$



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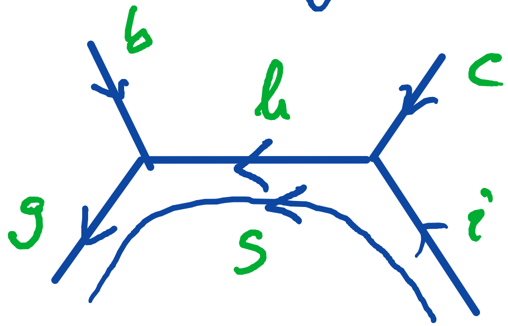
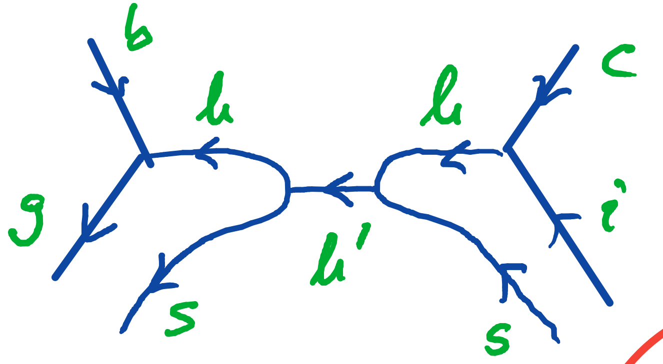
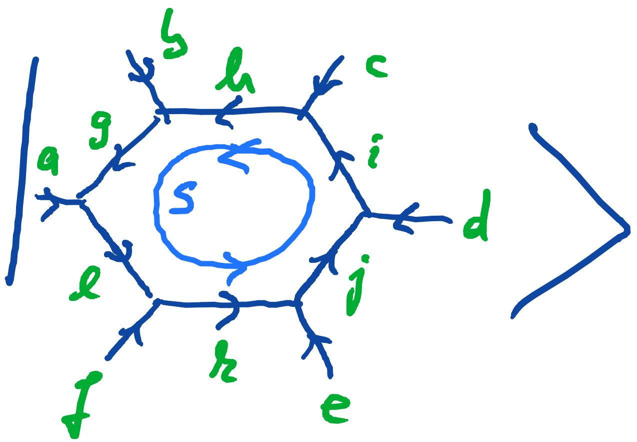


$$= \sum_{h'}$$



$$\Rightarrow$$



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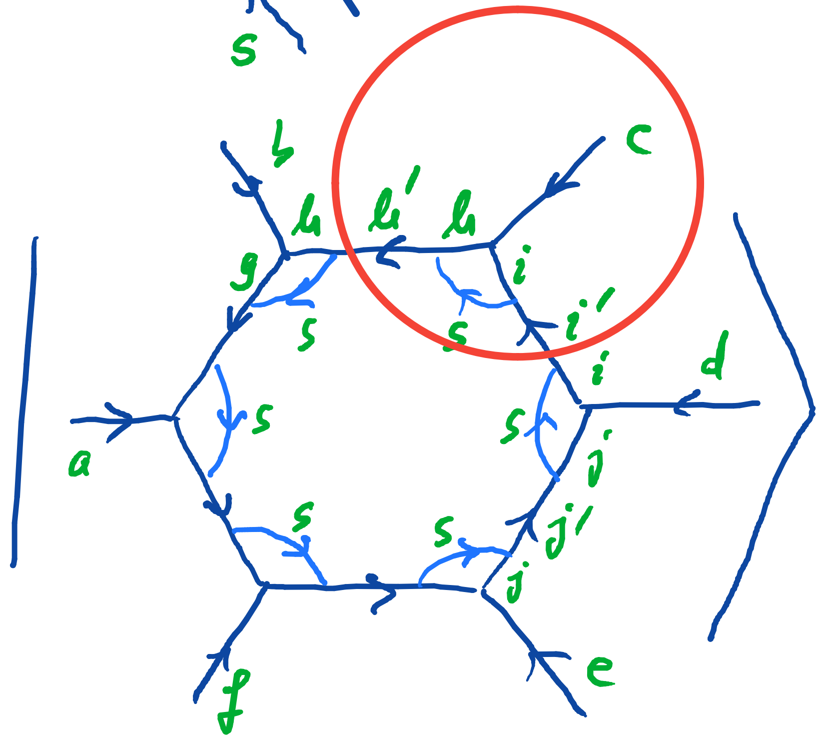
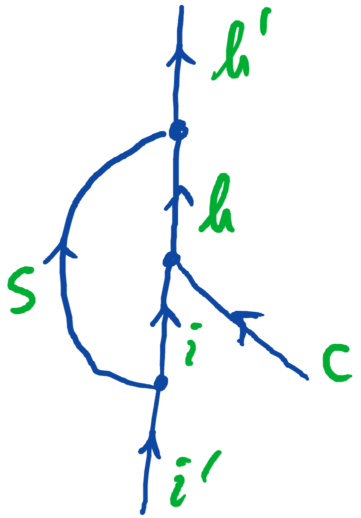


- **Now:**



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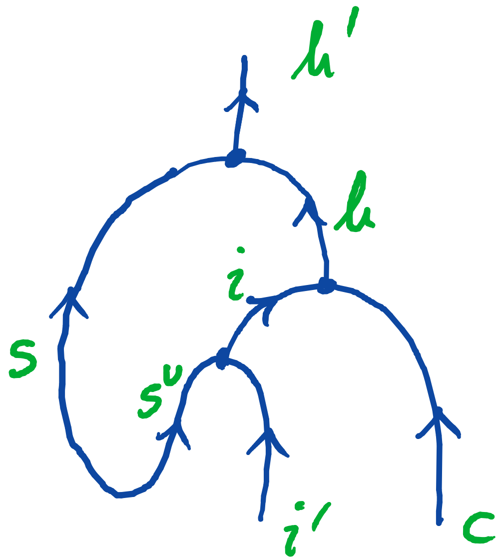

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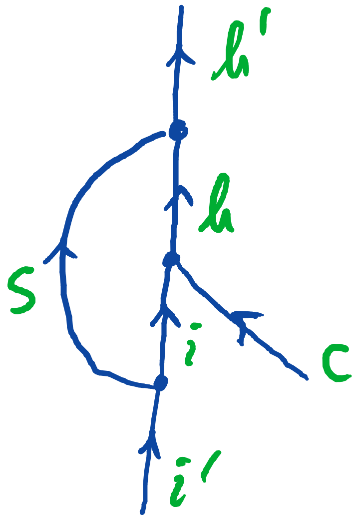

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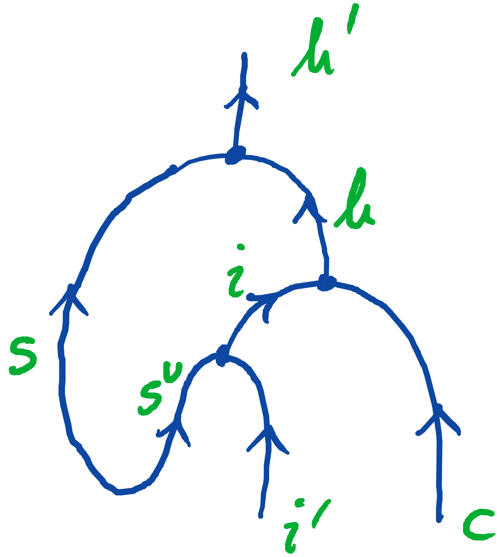
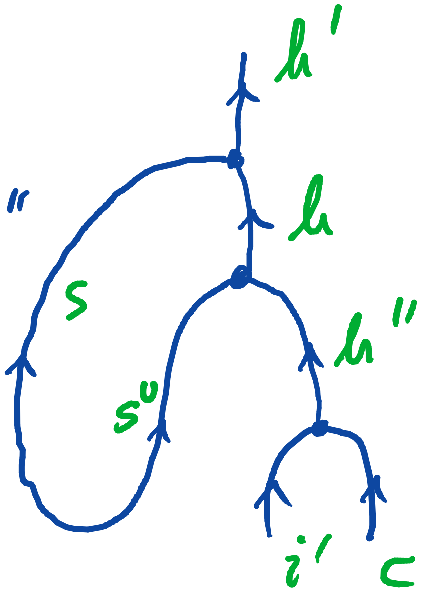


$$= \sum_{h''} \left(F^{h}_{s^\vee i' c}\right)_{i}^{h''}$$



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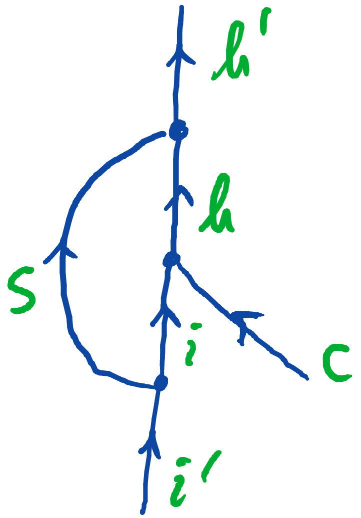

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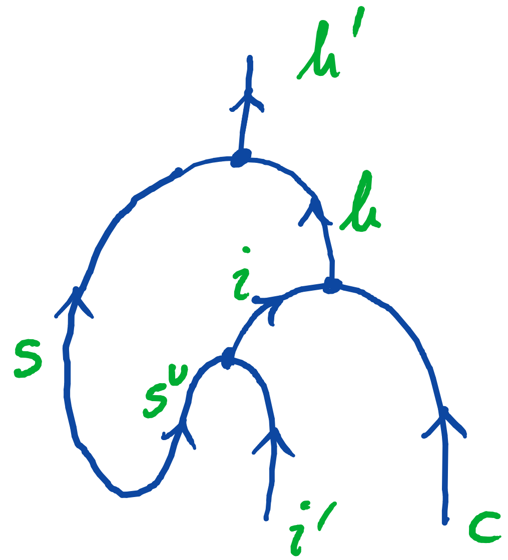

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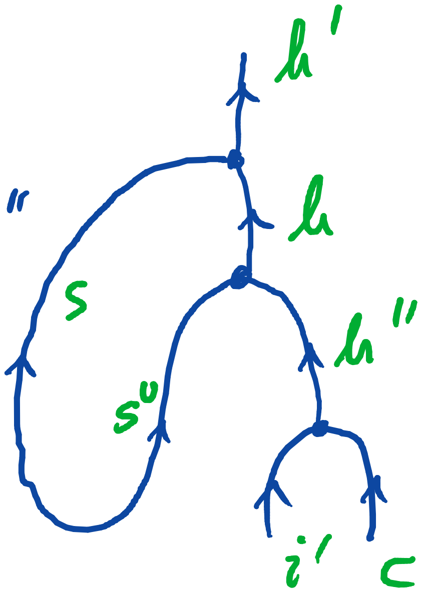

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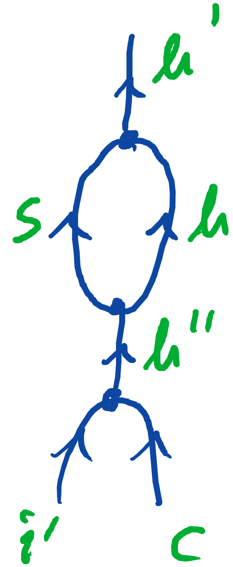

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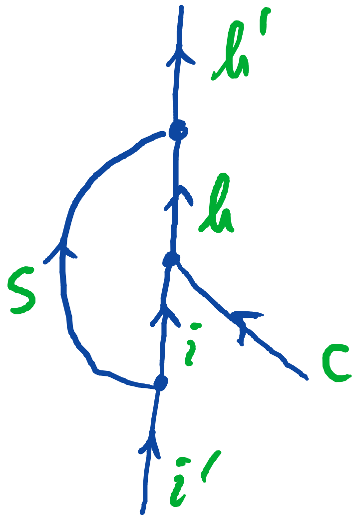

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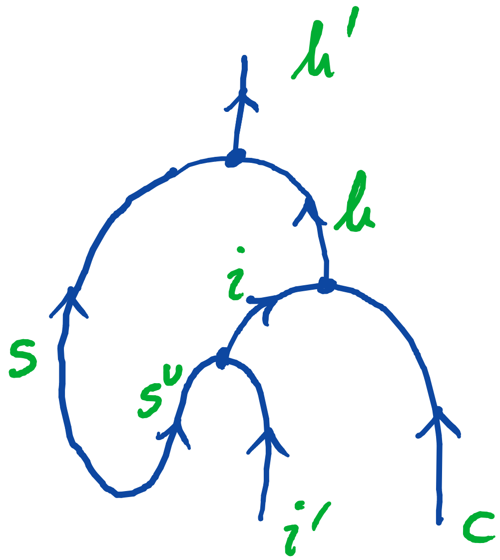

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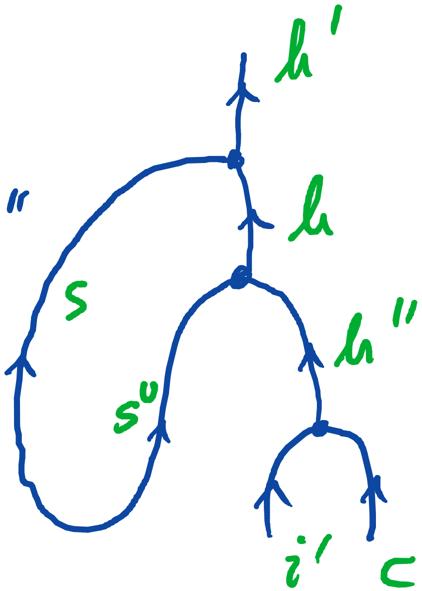

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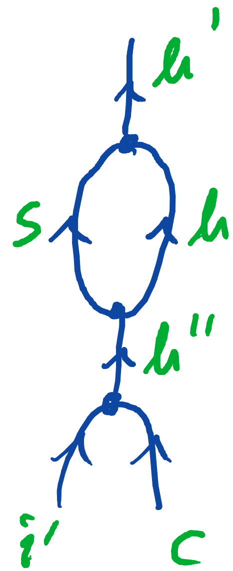

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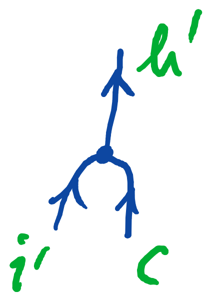
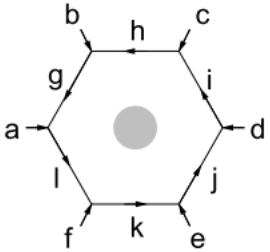
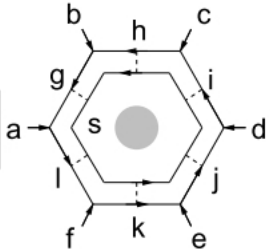
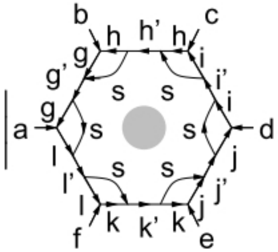


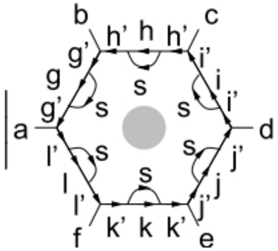
· Full computation (slightly different conventions):

$$B_{\boldsymbol{p}}^{s}\Big|$$


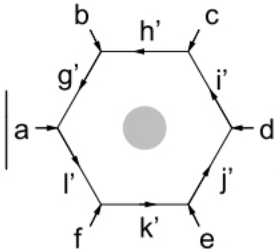
$$\Big\rangle = \Big|$$


$$\Big\rangle = \sum_{g'h'i'j'k'l'} F^{gg^*0}_{s^*sg'^*} F^{hh^*0}_{s^*sh'^*} F^{ii^*0}_{s^*si'^*} F^{jj^*0}_{s^*sj'^*} F^{kk^*0}_{s^*sk'^*} F^{ll^*0}_{s^*sl'^*} \Big|$$


$$\Big\rangle$$

$$= \sum_{g'h'i'j'k'l'} F^{gg^*0}_{s^*sg'^*} F^{hh^*0}_{s^*sh'^*} F^{ii^*0}_{s^*si'^*} F^{jj^*0}_{s^*sj'^*} F^{kk^*0}_{s^*sk'^*} F^{ll^*0}_{s^*sl'^*} F^{bg^*h}_{s^*h'g'^*} F^{ch^*i}_{s^*i'h'^*} F^{di^*j}_{s^*j'i'^*} F^{ej^*k}_{s^*k'j'^*} F^{fk^*l}_{s^*l'k'^*} F^{al^*g}_{s^*g'l'^*} \Big|$$


$$\Big\rangle$$

$$= \sum_{g'h'i'j'k'l'} F^{bg^*h}_{s^*h'g'^*} F^{ch^*i}_{s^*i'h'^*} F^{di^*j}_{s^*j'i'^*} F^{ej^*k}_{s^*k'j'^*} F^{fk^*l}_{s^*l'k'^*} F^{al^*g}_{s^*g'l'^*} \Big|$$


$$\Big\rangle \tag{C1}$$

This defines the action of B_p^s on arbitrary states.

• Properties of the Hamiltonian

• $[Q_v, Q_w] = [B_p^s, B_q^t] = [Q_v, B_p^s] = 0$

$\Rightarrow$ H is exactly solvable.

• Ground states $\Phi \in \mathcal{H}$ of H minimise each summand in H individually, i.e.

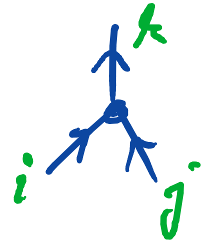
$$Q_v \Phi = \Phi \quad , \quad B_p^s \Phi = b_{p,+}^s \Phi \quad \forall\, v \in V,\ p \in F,\ s \in I,$$

highest eigenvalue of B_p^s.

• E.g.: For $\mathcal{C} = Rep_{\mathbb{Z}_2}$, H agrees with the Hamiltonian of the toric code on the honeycomb lattice.

• Ground States

• $Q_v \Phi = \Phi \quad \Rightarrow \quad \Phi$ is lin. comb. of states where each vertex



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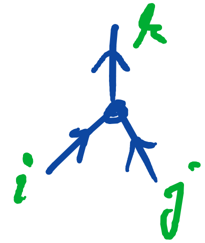

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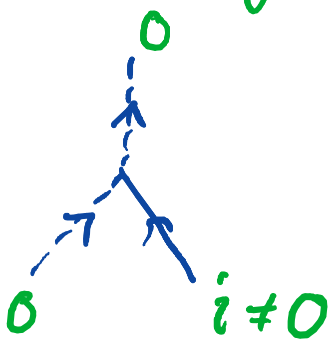
$\Rightarrow$ Strings in these states cannot have endpoints:



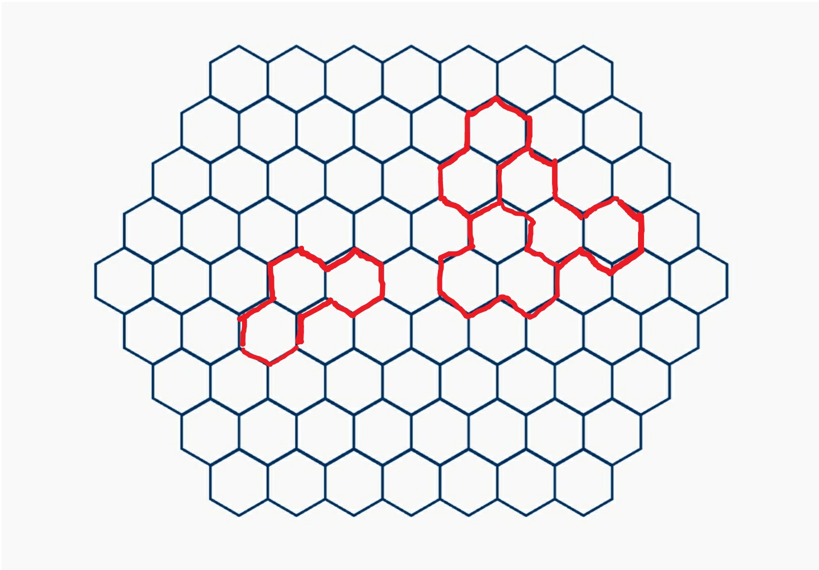
is never a valid branching, since

$$\mathbb{1} \otimes x_i \cong x_i \not\cong \mathbb{1} \quad \Rightarrow \quad N_{0i}^{0} = 0.$$

• Let $\mathcal{H}_Q := \ker\left(\mathbb{1} - \sum_{v \in V} Q_v\right)$

· Continuous representation of states

Lattice



$\mathcal{H}_Q$ spanned by :

e-labellings of edges of honeycomb lattice respecting branching rules

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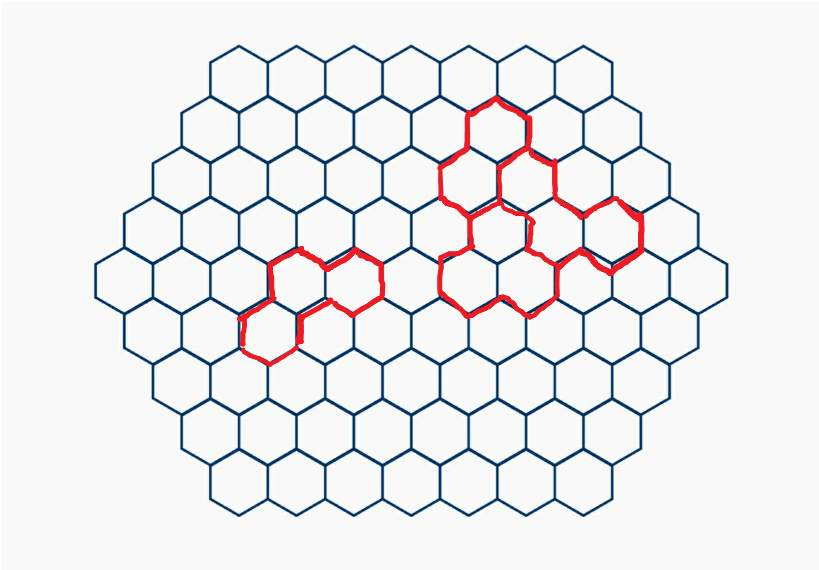
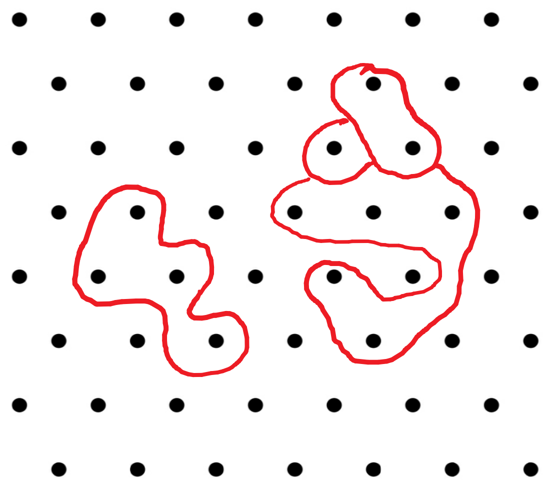


Thickened lattice



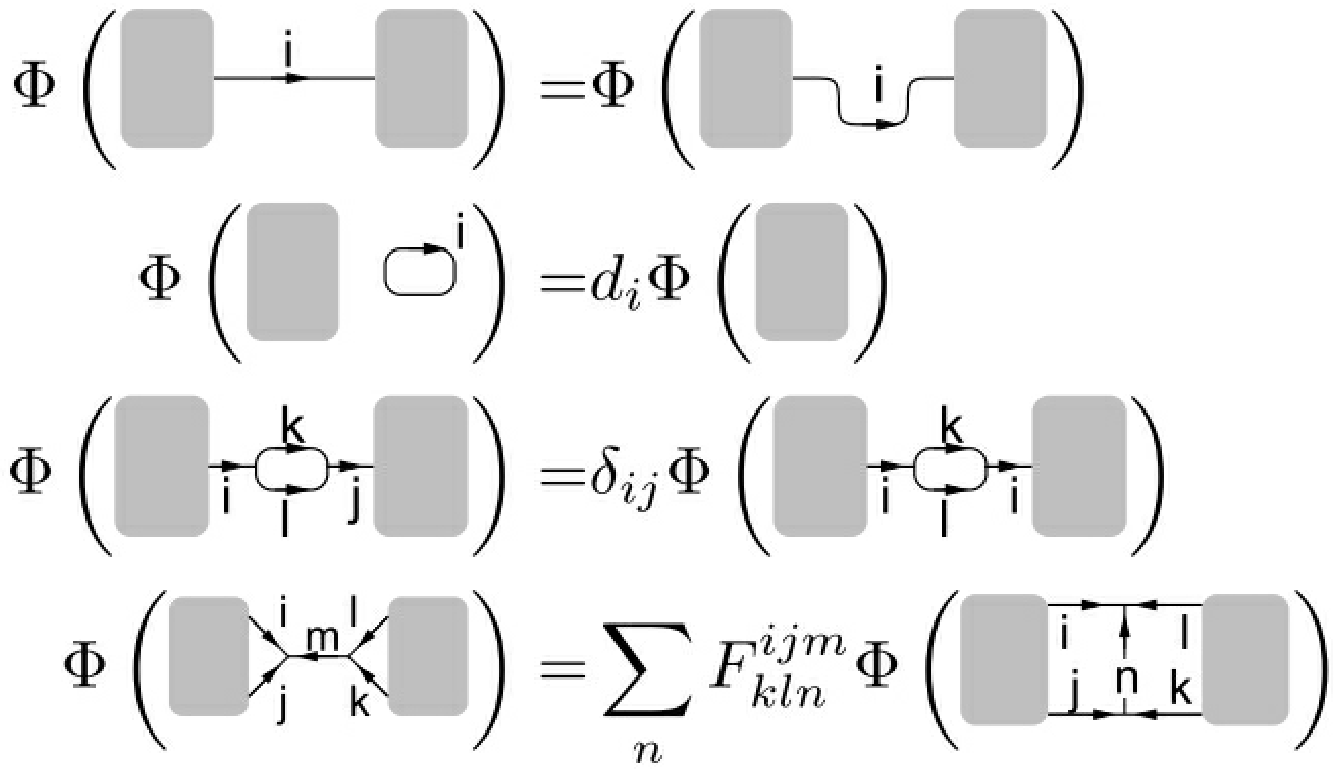
$\mathcal{H}_Q$ spanned by:

$\mathcal{C}$-labellings of edges of honeycomb lattice respecting branching rules

$\mathcal{C}$- labelled trivalent continuous, oriented graphs in surface $\setminus$ {centres of plaquettes} modulo local relations.

• Local Relations

• As long as no removed point is crossed, the local relations for states on the thickened lattice read as



(4) Isotopy

(5)

(6)

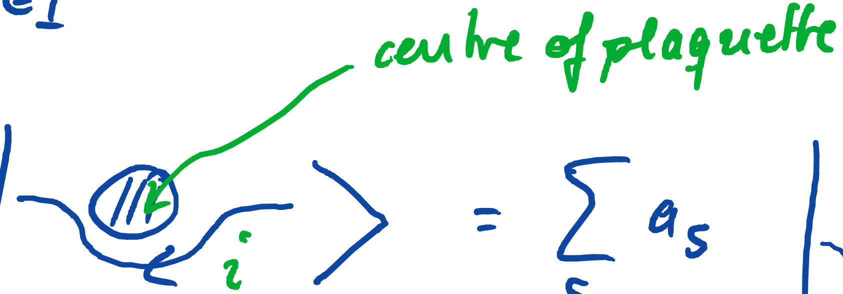
Renormalisation

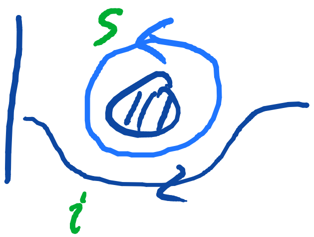
(7) CFT or uniqueness of ground st. from renorm.

$\rightsquigarrow$ This means we can use the string-diagram calculus of our spherical fusion cat $\mathcal{C}$ locally within $\mathcal{H}_Q$.

↳ General + rigorous picture: Yang Yang's talk.

- Recall that $B_p := \sum_{s \in I} a_s \cdot B_p^s$.

- We compute: $B_p \Big|$ 

$\Big\rangle = \sum_s a_s \Big|$ 

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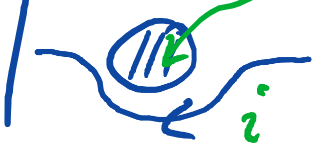
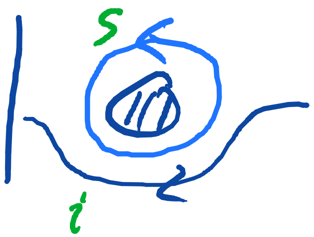
- We compute: $B_p \Big|$ 

$\Big\rangle = \sum_s a_s \Big|$ 

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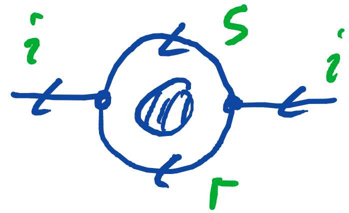
$= \sum_{s,r} a_s \cdot \Big|$ 

$\Big\rangle \quad = \sum_{s,r} a_s \Big|$ 

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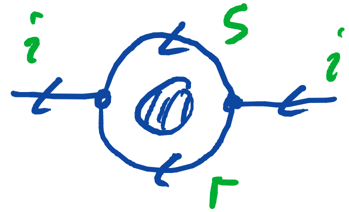
$\Big\rangle = \sum_{s,r} a_s \Big|$ 

$\Big\rangle$

for good choice of a_s $\Rightarrow$ $= B_p \Big|$ 

$\Big\rangle$

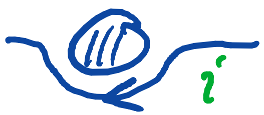
- Levin-Wen: There exists a choice of the a_s s.t. ground states satisfy $B_p \Phi = \Phi$.

$\Rightarrow$ For such a state, we obtain:

$0 = \langle$ 

$| B_p | \Phi \rangle - \langle$ 

$| B_p | \Phi \rangle$

$= \Phi\Big($ 

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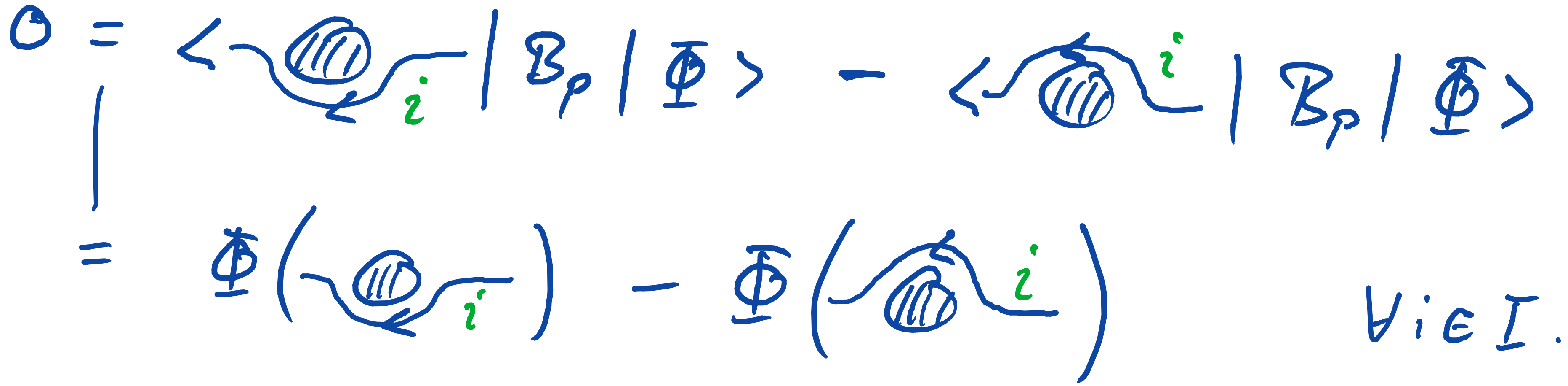


$\Rightarrow$ Strings making up Φ can be isotoped across the removed points $\Rightarrow$ Φ satisfies the local relations globally!

$\Downarrow$

↳ Motivation for Yang Yang's talk. "topological state"

• Excitations and Quasiparticles

- We work with the special values for the a_s.

  $\rightsquigarrow$ $Q_v \Phi = \Phi = B_p \Phi \quad \forall v \in V, p \in F$ on gd. state.

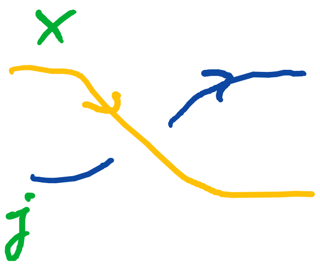
- QP excitations: states that violate these constraints only locally.

- Idea: Insert (superposition of) strings with labels $i \in I$ along paths $P \subset E$ in the thickened lattice and procede like for B_p^s to reduce to lattice state.

- In particular, a quasiparticle is a superposition

$$\xrightarrow{x} \;=\; \sum_{i \in I} n_{x,i} \xrightarrow{i} \quad , \quad n_{x,i} \in \mathbb{N}_0$$

- In order to allow for non-trivial statistics, we allow non-trivial crossings:



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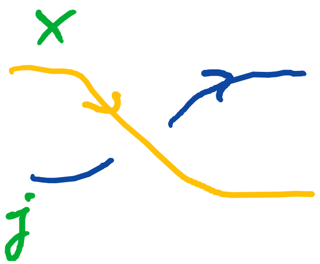

in string-diag interpretation:

$$\begin{tikzcd} X\otimes X_j \arrow[r] \arrow[d, equal] & X_j \otimes X \arrow[d, equal] \\ \sum_i n_{x,i}\, X_i\otimes X_j \arrow[r, dashed] & \sum_k n_{x,k}\, X_j \otimes X_k \end{tikzcd}$$

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$$\xrightarrow{x} = \sum_{i\in I} \mu_{x,i} \xrightarrow{i} , \quad \mu_{x,i} \in \mathbb{N}_0$$

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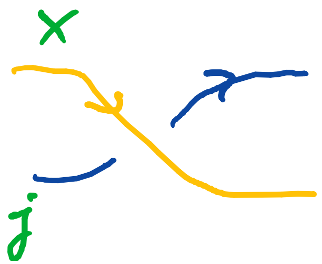
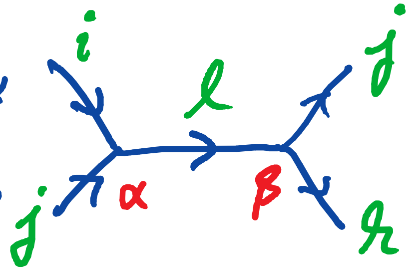
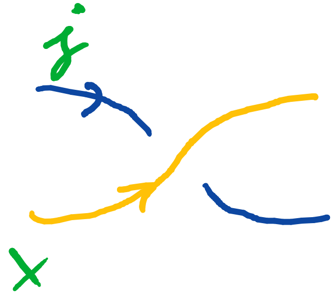


in string-diag interpretation: $x \otimes x_j \longrightarrow x_j \otimes x$

$$\sum_i \mu_{x,i}\, x_i \otimes x_j \dashrightarrow \sum_k \mu_{x,k}\, x_j \otimes x_k$$

$$= \sum_{\ell,\alpha,\beta} \left(\Omega^{k}_{x,ij\,\ell}\right)^{\alpha}_{\beta}$$

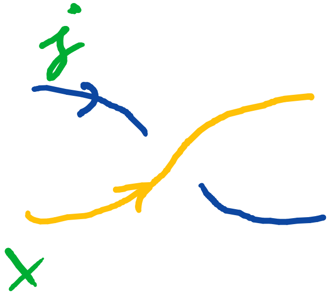


- Similarly : $\bar{\Omega}_x$ for 

- QPs determined by $(u_{x,i}\,,\ \Omega_x,\ \bar{\Omega}_x) =: \hat{x}$

- $W_{\hat{x}}(P)$:= operator creating QP $\hat{x}$ along path P.

- $[W_{\hat{x}}(P),\ B_p^s] = 0$

- Similarly: $\bar{\Omega}_x$ for 

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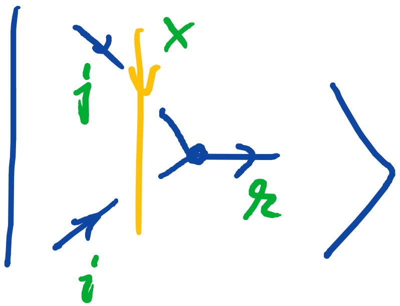
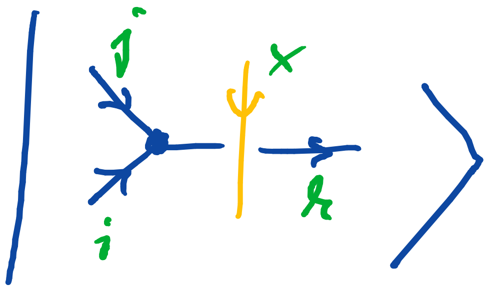
$\Leftrightarrow$ 

$=$ 

"$\Omega_x, \bar{\Omega}_x$ monoidal"

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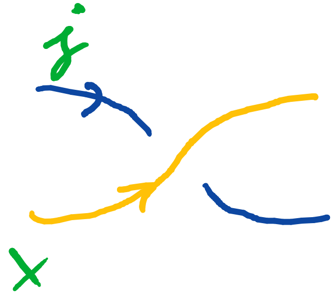

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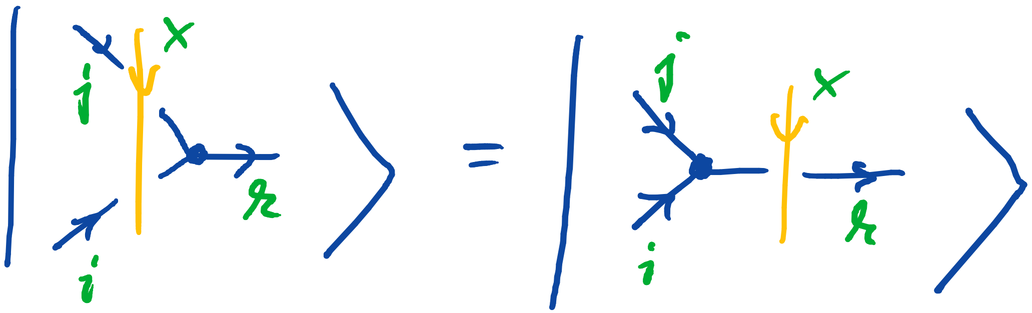

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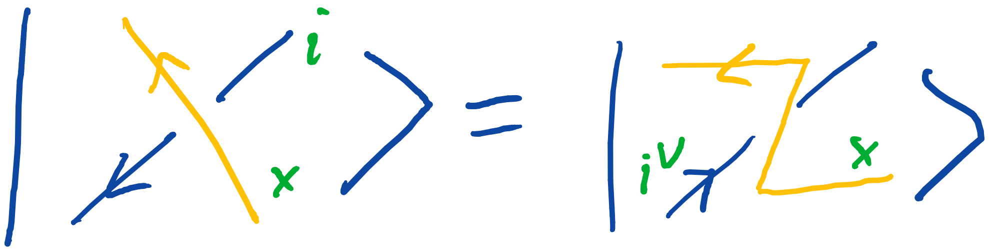

relates Ω_x and $\bar{\Omega}_x$.

$(\bar{\Omega}_x = \Omega_x^{-1})$

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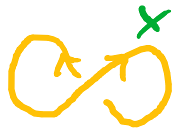
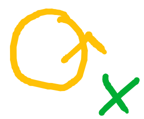
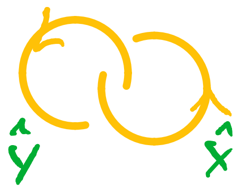




and their S-matrix

$S_{\hat{x},\hat{y}} = \langle \Phi | \text{ } | \Phi \rangle$



• Concluding Remarks

- LW models admit QPs with non-trivial statistics (e.g. $\mathcal{C}$ = Fibonacci cat $\rightsquigarrow$ universal QComp!)
- Can describe a wide variety of (all?) topological phases
- Intricately related to TQFTs through their input
- Mathematical formulation: Kirillov (next talk)