Homotopical computations for Abelian gauge theory

Based on joint work with A. Schenkel and R.J. Szabo.

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Outline

1. Sketching the strategy

2. Configurations first…
   - a The recipe to cook a tasty homotopy limit
   - b The diagram for a “canonical” cover
   - c The result of the computation
   - d Reproducing all bundle-connection pairs & functor extension

3. …then observables
   - a The recipe to cook a tasty homotopy colimit
   - b The diagram for a “canonical” cover
   - c The result of the computation
   - d Detecting all bundle-connection pairs up to gauge & functor extension

Our status with Abelian gauge theory: Configurations

- **Configurations** on U contractible form a nice **action groupoid**:

$$\Omega^1(U;\mathfrak{g}) \leftleftarrows C^\infty(U;G)\times\Omega^1(U;\mathfrak{g});$$

- Equivalently, we consider the corresponding simplicial set:

$$\Omega^1(U;\mathfrak{g}) \leftleftarrows C^\infty(U;G)\times\Omega^1(U;\mathfrak{g}) \,\overleftarrow{\overleftarrow{\overleftarrow{}}}\, C^\infty(U;G)^{\times 2}\times\Omega^1(U;\mathfrak{g}) \,\overleftarrow{\overleftarrow{\overleftarrow{\overleftarrow{}}}}\, \cdots$$

Huge simplification: Since $G = U(1)$ is Abelian, this is actually a **simplicial Abelian group**.

- Taking the normalized Moore complex, we get a non-negatively graded **chain complex** of Abelian groups:

$$\begin{aligned}\mathfrak{C}(U): 0 \longleftarrow \Omega^1(U;\mathfrak{g})_0 &\xleftarrow{\delta} C^\infty(U;G)_1 \longleftarrow 0,\\ g\,\mathrm{d}\,g^{-1} &\longleftarrow\!\shortmid\; g.\end{aligned}$$

Our status with Abelian gauge theory: Observables

- The advantage of the Abelian case is that we can take **observables as "smooth" group characters**:

$$\chi \oplus \varphi \in \Omega^m_{c,\mathbb{Z}}(U; \mathfrak{g}^*)_{-1} \oplus \Omega^{m-1}_c(U; \mathfrak{g}^*)_0,$$
$$A \oplus g \in \Omega^1(U; \mathfrak{g})_0 \oplus C^\infty(U; G)_1,$$
$$\langle \chi \oplus \varphi, A \oplus g \rangle = \exp\Big(\int_U (A \wedge \varphi + \log(g)\chi) \Big).$$

- Observables form a non-positively graded **chain complex** of Abelian groups, dual to the one for configurations:

$$\mathfrak{O}(U) : 0 \longleftarrow \Omega^m_{c,\mathbb{Z}}(U; \mathfrak{g}^*)_{-1} \xleftarrow{\delta^*} \Omega^{m-1}_c(U; \mathfrak{g}^*)_0 \longleftarrow 0,$$
$$\mathrm{d}\,\varphi \longleftarrow\!\shortmid\, \varphi.$$

A basic observation

Consider all (m-dimensional, oriented) contractible manifolds U. This defines the full subcategory $\mathsf{Man}_{\copyright}$ of the category Man of (m-dimensional, oriented) manifolds with open embeddings.

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Our chain complexes of configurations and observables are functorially assigned to contractible manifolds:

$$\mathfrak{C} : \mathsf{Man}_{©}^{\mathrm{op}} \longrightarrow \mathsf{Ch}_{\geq 0}(\mathsf{Ab}), \qquad \mathfrak{O} : \mathsf{Man}_{©} \longrightarrow \mathsf{Ch}_{\leq 0}(\mathsf{Ab}).$$

In fact, given an open embedding $f : U \to V$:

$$\mathfrak{C}(f) : A_V \oplus g_V \longmapsto f^* A_V \oplus f^* g_V, \qquad \mathfrak{O}(f) : \chi_U \oplus \varphi_U \longmapsto f_* \chi_U \oplus f_* \varphi_U.$$

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We will compute the homotopy (co)limits of certain diagrams arising from those functors.

The "canonical" cover

Take a manifold M and consider all its contractible open subsets U. Taking as arrows only the inclusions between two subsets, $U \subseteq V$, we get a category $\mathsf{D}(M)$.

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Some remarks:

- The set of objects of $\mathsf{D}(M)$ is a cover of M;
- $\mathsf{D}(M)$ is a subcategory of $\mathsf{Man}_{©}$;
- An open embedding $f : M \to M'$ induces $\mathsf{D}(f) : \mathsf{D}(M) \to \mathsf{D}(M')$, sending $U \subseteq V$ in M to $f(U) \subseteq f(V)$ in M'.

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Recall that we want to **glue data on trivial regions to get global data**. Since the objects of $\mathrm{D}(M)$ cover M, $\mathrm{D}(M)$ is a good candidate for assigning local data. Even more, $\mathrm{D}(M)$ is functorial in M, inducing a functorial behavior of our homotopy (co)limits with respect to open embeddings $f : M \to M'$.

The diagram for configurations

$$\mathfrak{C}_M = \mathfrak{C}|_{\mathsf{D}(M)} : \mathsf{D}(M)^{\mathrm{op}} \longrightarrow \mathsf{Ch}_{\geq 0}(\mathsf{Ab}).$$

$\mathsf{D}(M)$ provides the "shape", while $\mathfrak{C} : \mathsf{Man}^{\mathrm{op}}_{©} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ gives to the diagram its actual content.

Meaning: For a fixed manifold M, look only at the configurations on its contractible subsets (those are easy to handle).

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Naturality: From an open embedding $f : M \to M'$, we get a natural isomorphism $\mathfrak{C}_f : \mathfrak{C}_{M'} \circ \mathsf{D}(f) \to \mathfrak{C}_M$. This property is inherited from our choice of the "canonical" cover.

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We try to recover fully the configurations on M by computing the homotopy limit of the diagram $\mathfrak{C}_M : \mathsf{D}(M)^{\mathrm{op}} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$.

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We try to recover global observables by computing the homotopy colimit of the diagram $\mathfrak{O}_M : \mathsf{D}(M) \to \mathsf{Ch}_{\leq 0}(\mathsf{Ab})$.

Goals

Configurations: Given a manifold M, compute

$$\mathfrak{C}_{\mathsf{ext}}(M) := \underline{\mathrm{ho}}\lim \left(\mathfrak{C}_M : \mathsf{D}(M)^{\mathrm{op}} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})\right),$$

and check that:

1. $\mathfrak{C}_{\mathsf{ext}}$ extends $\mathfrak{C}$ from $\mathsf{Man}_{©}^{\mathrm{op}}$ to $\mathsf{Man}^{\mathrm{op}}$ (up to natural weak equiv.);
2. $\mathfrak{C}_{\mathsf{ext}}(M)$ is weakly equivalent to the Deligne complex for bundles with connection on M.

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Observables: Compute

$$\mathfrak{O}_{\mathsf{ext}}(M) := \underline{\mathrm{ho}}\mathrm{colim}\left(\mathfrak{O}_M : \mathsf{D}(M) \to \mathsf{Ch}_{\leq 0}(\mathsf{Ab})\right),$$

and check that

1. $\mathfrak{O}_{\mathsf{ext}}$ extends $\mathfrak{O}$ from $\mathsf{Man}_{©}$ to Man (up to a natural weak equiv.);
2. The pairing between $\mathfrak{O}_{\mathsf{ext}}(M)$ and $\mathfrak{C}_{\mathsf{ext}}(M)$ separates $\mathfrak{C}_{\mathsf{ext}}(M)$.

A recipe to cook $\underline{\mathrm{ho}}\mathrm{lim}$ See Dugger and Rodríguez-González

1. Take the cosimplicial replacement of $\mathfrak{C}_M$, an object of $\mathrm{cCh}_{\geq 0}(\mathrm{Ab})$:

$$\prod_U \mathfrak{C}_M(U) \rightrightarrows \prod_{U\subseteq V} \mathfrak{C}_M(U) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{U\subseteq V\subseteq W} \mathfrak{C}_M(U) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots.$$

2. Produce the co-normalized Moore complex in $\mathrm{Ch}_{\leq 0}(\mathrm{Ch}_{\geq 0}(\mathrm{Ab}))$:

$$\mathfrak{C}_{M0,*} = \prod_U \mathfrak{C}_M(U)_*, \qquad \mathfrak{C}_{M-n,*} = \prod_{U_0\subset\ldots\subset U_n} \mathfrak{C}_M(U_0)_*.$$

3. Take the $\prod$-total complex in $\mathrm{Ch}(\mathrm{Ab})$ of this double complex:

$$\mathfrak{C}_M^{\mathrm{tot}} = \Big(\bigoplus_{n\in\mathbb{Z}} \prod_{p+q=n} \mathfrak{C}_{M\,p,q},\ \delta^{\mathrm{tot}} = \delta^{\mathrm{v}} + (-1)^p \delta^{\mathrm{h}} \Big).$$

4. Truncation of $\mathfrak{C}_M^{\mathrm{tot}}$ provides $\mathfrak{C}_{\mathrm{ext}}(M) := \underline{\mathrm{ho}}\mathrm{lim}\, \mathfrak{C}_M$ in $\mathrm{Ch}_{\geq 0}(\mathrm{Ab})$.

The output: Global configurations

Following the recipe, we get $\mathfrak{C}_{\mathrm{ext}}(M)$ in $\mathrm{Ch}_{\geq 0}(\mathrm{Ab})$:

$$0 \longleftarrow \mathfrak{C}_{\mathrm{ext}}(M)_0 \xleftarrow{\ \delta\ } \mathfrak{C}_{\mathrm{ext}}(M)_1 \longleftarrow 0,$$

$$\mathfrak{C}_{\mathrm{ext}}(M)_1 = \prod_{U} C^{\infty}(U; G), \quad \mathfrak{C}_{\mathrm{ext}}(M)_0 \subseteq \prod_{U} \Omega^1(U; \mathfrak{g}) \times \prod_{U \subset V} C^{\infty}(U; G).$$

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$\prod_U A_U \times \prod_{U\subset V} g_{(U\subset V)}$ is in $\mathfrak{C}_{\mathrm{ext}}(M)_0$ iff

$$A_V|_U - A_U = g_{(U\subset V)}\,\mathrm{d}\,g^{-1}_{(U\subset V)}, \qquad \forall\, U\subset V,$$
$$g_{(V\subset W)}|_U\, g^{-1}_{(U\subset W)}\, g_{(U\subset V)} = 1, \qquad \forall\, U\subset V\subset W.$$

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Finally, the differential is specified by

$$\delta \prod_U g_U = \prod_U g_U \, \mathrm{d} \, g_U^{-1} \times \prod_{U \subset V} g_V|_U \, g_U^{-1}.$$

$\mathfrak{C}_{\mathsf{ext}}(M)$ classifies bundle-connection pairs

Homology of $\mathfrak{C}_{\mathsf{ext}}(M)$:

$$\mathrm{H}_*(\mathfrak{C}_{\mathsf{ext}}(M)) = \hat{\mathrm{H}}^2(M;\mathbb{Z}) \oplus \mathrm{H}^0(M, G).$$

To compute it, compare to the Deligne complex associated to $\mathrm{D}(M)$.

The homology of $\mathfrak{C}_{\mathsf{ext}}(M)$ reproduces all bundles with connections in degree 0 and gives back the global gauge group in degree 1.

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Remarks:

- $\mathfrak{C}_{\mathsf{ext}}$ is a functor from $\mathsf{Man}^{\mathrm{op}}$ to $\mathsf{Ch}_{\geq 0}(\mathsf{Ab})$;
- $\mathfrak{C}_{\mathsf{ext}}|_{\mathsf{Man}_{©}}$ is naturally weakly equivalent to $\mathfrak{C}$.

$\mathfrak{C}_{\mathsf{ext}} : \mathsf{Man}^{\mathrm{op}} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ is a functor extension (up to weak equivalence) of $\mathfrak{C} : \mathsf{Man}_{©}^{\mathrm{op}} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$.

A recipe to cook $\underline{\mathrm{ho}}\mathrm{colim}$ See Dugger and Rodríguez-González

1. Take the simplicial replacement of $\mathfrak{O}_M$, an object of $\mathrm{sCh}_{\leq 0}(\mathrm{Ab})$:

$$\coprod_U \mathfrak{O}_M(U) \leftleftarrows \coprod_{U\subseteq V} \mathfrak{O}_M(U) \Lleftarrow \coprod_{U\subseteq V\subseteq W} \mathfrak{O}_M(U) \cdots .$$

2. Produce the normalized Moore complex in $\mathrm{Ch}_{\geq 0}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))$:

$$\mathfrak{O}_{M\,0,*} = \coprod_U \mathfrak{O}_M(U)_*, \qquad \mathfrak{O}_{M\,n,*} = \coprod_{U_0\subset\ldots\subset U_n} \mathfrak{O}_M(U_0)_*.$$

3. Take the $\coprod$-total complex in $\mathrm{Ch}(\mathrm{Ab})$ of this double complex:

$$\mathfrak{O}_M^{\mathrm{tot}} = \Big(\bigoplus_{n\in\mathbb{Z}} \coprod_{p+q=n} \mathfrak{O}_{M\,p,q},\ \delta^{\mathrm{tot}} = \delta^{\mathrm{v}} + (-1)^p \delta^{\mathrm{h}} \Big).$$

4. Truncation of $\mathfrak{O}_M^{\mathrm{tot}}$ provides $\mathfrak{O}_{\mathrm{ext}}(M) := \underline{\mathrm{ho}}\mathrm{colim}\,\mathfrak{O}_M$ in $\mathrm{Ch}_{\leq 0}(\mathrm{Ab})$.

1. Simplicial replacement

Recall that $\mathfrak{O}_M : \mathsf{D}(M) \to \mathsf{Ch}_{\leq 0}(\mathsf{Ab})$ on objects is

$$\begin{gathered} 0 \longleftarrow \Omega^m_{c,\mathbb{Z}}(U;\mathfrak{g}^*)_{-1} \xleftarrow{\delta^*} \Omega^{m-1}_c(U;\mathfrak{g}^*)_0 \longleftarrow 0, \\ \mathrm{d}\,\varphi \longleftarrow\!\shortmid \varphi. \end{gathered}$$

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We take the **nerve** of $\mathsf{D}(M)$, i.e. the simplicial set outlined below:

0 : objs. U, 1 : morphs. $U \subseteq V$, 2 : 2 comp. morphs. $U \subseteq V \subseteq W$, ...

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0 : objs. U, 1 : morphs. $U \subseteq V$, 2 : 2 comp. morphs. $U \subseteq V \subseteq W$, ...

We use the nerve to "label" coproducts, so we get a simplicial object in chain complexes, i.e. an object of $\mathsf{sCh}_{\leq 0}(\mathsf{Ab})$:

$$\coprod_{U} \mathfrak{O}_M(U) \leftleftarrows \coprod_{U\subseteq V} \mathfrak{O}_M(U) \Lleftarrow \coprod_{U\subseteq V\subseteq W} \mathfrak{O}_M(U) \cdots$$

Notice that, on each level, the differentials are specified by the universal property of the coproduct.

2. Normalized Moore complex

At each level of the simplicial replacement, quotient by the images of degeneracy maps. This removes $U = U$ from coproducts:

$$\mathfrak{O}_{M0,*} = \coprod_{U} \mathfrak{O}_M(U)_*, \qquad \mathfrak{O}_{Mn,*} = \coprod_{U_0 \subset \ldots \subset U_n} \mathfrak{O}_M(U_0)_*.$$

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Take the alternating sum of face maps to define the vertical differential: $\delta^v = \sum_{i=0}^{n} \partial_i^n$. We get a chain complex of chain complexes, more precisely, an object in $\mathrm{Ch}_{\geq 0}\big(\mathrm{Ch}_{\leq 0}(\mathrm{Ab})\big)$:

$$0 \longleftarrow \mathfrak{O}_{M0,*} \xleftarrow{\delta^v} \mathfrak{O}_{M1,*} \xleftarrow{\delta^v} \mathfrak{O}_{M2,*} \xleftarrow{\delta^v} \ldots$$

At each non-neg. degree, a chain complex in non-pos. degree!

3. $\coprod$-total complex

We have a double complex:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\Big\downarrow{\scriptstyle \delta^v} & & \Big\downarrow{\scriptstyle \delta^v} \\
\left(\coprod_{U \subset V} \Omega^m_{c,\mathbb{Z}}(U; \mathfrak{g}^*) \right)_{-1,1} & \xleftarrow{\ \delta^h\ } & \left(\coprod_{U \subset V} \Omega^{m-1}_c(U; \mathfrak{g}^*) \right)_{0,1} \\
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\end{array}
$$

and we take its coproduct totalization, which is in degrees ≥ -1:

$$
\coprod_U \Omega^m_{c,\mathbb{Z}}(U; \mathfrak{g}^*) \overset{\delta^v + \delta^h}{\longleftarrow} \coprod_U \Omega^{m-1}_c(U; \mathfrak{g}^*) \oplus \coprod_{U \subset V} \Omega^m_{c,\mathbb{Z}}(U; \mathfrak{g}^*) \overset{\delta^v - \delta^h}{\longleftarrow} \dots
$$

4. Truncation

The total complex is not in $\mathrm{Ch}_{\leq 0}(\mathrm{Ab})$, so we have to truncate:

$$\mathfrak{O}_{\mathrm{ext}}(M) : \coprod_U \Omega^m_{c,\mathbb{Z}}(U;\mathfrak{g}^*) \xleftarrow{\delta^v + \delta^h} \frac{\coprod_U \Omega^{m-1}_c(U;\mathfrak{g}^*) \oplus \coprod_{U\subset V} \Omega^m_{c,\mathbb{Z}}(U;\mathfrak{g}^*)}{\mathrm{im}(\delta^v - \delta^h)}.$$

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The extended pairing on M in degree 0, 0

A global configuration in degree 0:

$$C_0 = \prod_U A_U \times \prod_{U \subset V} g_{(U \subset V)} \in \prod_U \Omega^1(U; \mathfrak{g}) \times \prod_{U \subset V} C^\infty(U; G),$$

subject to constraints...

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A global observable in degree 0:

$$O_0 = \coprod_U \varphi_U \oplus \coprod_{U \subset V} \chi_{U \subset V} \in \coprod_U \Omega_c^{m-1}(U; \mathfrak{g}^*) \oplus \coprod_{U \subset V} \Omega^m_{c,\mathbb{Z}}(U; \mathfrak{g}^*),$$

up to identifications...

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up to identifications...

$$\langle O_0, C_0 \rangle^{\mathrm{ext}} = \exp\Big(\sum_U \int_U A_U \wedge \varphi_U - \sum_{U \subset V} \int_U \log(g_{U \subset V})\, \chi_{U \subset V} \Big).$$

Remark: $\langle O_0, C_0 \rangle^{\mathrm{ext}} = 1$ for all O_0 entails that C_0 is trivial!

The global pairing: -1, 1

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A global configuration in degree 1:

$$C_1 = \prod_U g_U \in \prod_U C^\infty(U; G).$$

A global observable in degree -1:

$$O_{-1} = \coprod_U \chi_U \in \coprod_U \Omega^m_{c,\mathbb{Z}}(U; \mathfrak{g}^*).$$

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$$C_1 = \prod_U g_U \in \prod_U C^\infty(U; G).$$

A global observable in degree -1:

$$O_{-1} = \prod_U \chi_U \in \prod_U \Omega^m_{c,\mathbb{Z}}(U; \mathfrak{g}^*).$$

The pairing:

$$\langle O_{-1}, C_1 \rangle = \exp\Big(\sum_U \int_U \log(g_U)\, \chi_U \Big).$$

Remark: $\langle O_{-1}, C_1 \rangle^{\text{ext}} = 1$ for all O_{-1} entails that C_1 is trivial!

Conclusions and outlook

- For gauge theories, we propose to obtain global observables by means of $\underline{\text{ho}}\text{colim}$, i.e. adapting by means of homotopical algebra the idea of a universal algebra for gauge theory;
- We test this approach in a simplified setting: Abelian gauge theory, no dynamics, no quantization. Observables obtained via $\underline{\text{ho}}\text{colim}$ can detect all $U(1)$-bundles with connection!

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Thank you very much for your attention!